End Semester Examination
Electrodynamics

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Duration: 180 minutes,
Total points: 100.

Please give arguments where necessary. If it is unclear from your answer why a particular step is being taken, full credit will not be awarded. Grades will be awarded not only based on what final answer you get, but also on the intermediate steps.

1. A point dipole $\vec{p} = p\hat{e_z}$ is located at the center of a spherical cavity of radius a. The interior of the cavity is free space, but the outside is a simple dielectric of permittivity ϵ.

   (a) Find the electric potential ϕ and the electric field $\vec{E}$ everywhere. (*While calculating the form of the potential at the interior, do remember that the potential from the dipole itself diverges at the origin, but the potential from any other source should not. Keep this in mind if you plan to do an expansion.*)

   (b) Show that, in the limit $\epsilon \to \infty$, the tangential component of the electric field at the interface of the dielectric with free space vanishes (this means the interface becomes an equipotential and the effect of the dielectric in the interior is as if the dielectric was a perfect conductor). Find the potential in the cavity in this limit.

   (c) In the previous part, you derived the expression for the interior potential in the limit $\epsilon \to \infty$. Show that the same expression is obtained if you solve for the **interior potential** by taking the interface to be an equipotential as your boundary condition from the very beginning i.e., if you do the first part with this boundary condition.

   10 + 4 + 6 = 20 points.

2. A sphere made of an Ohmic conductor of uniform conductivity σ_b and radius R is immersed in another Ohmic conductor of uniform conductivity σ_a. Assume the second conductor fills all space outside the first. Two

electrodes at infinity impose a uniform current density J_0. After equilibrium is reached and we reach steady current condition, find the electric potential everywhere.
10 points

3. An infinite cylinder of radius R is extended along the z-axis and carries is a steady current density $\vec{j}(\vec{r}) = j_0 \frac{\rho}{R} \hat{e_z}$. There is no current outside the cylinder. Define the z-component of the vector potential to be zero at the origin.

   (a) Find the magnetic field everywhere.

   (b) Show that if the azimuthal component of the magnetic field has to be finite at $\rho = R$, A_z has to be continuous there.

   (c) Find $\vec{A}$ everywhere.

   (d) Consider a vertical rectangular loop with length L along the z-axis. The inner vertical leg is at $\rho = 0$ and the outer vertical leg is at $\rho = a$, $a > R$. Calculate the flux through the loop using

   - the magnetic field from part (a).
   - the vector potential from part (c).

   Show that the two answers match.

   5 + 3 + 7 + (2 + 3) = 20 points

4. (a) In Cartesian co-ordinates, a current density along a given Cartesian axis will produce a vector potential in that direction alone, at least in the Coulomb gauge (this is guaranteed by the vector Poisson equation, as you know). However, this conclusion is not generally true in other co-ordinate systems, except in certain special cases. The first question is about one such special case: show that an azimuthally symmetric current density in the $\hat{e_\phi}$ direction produces an azimuthally symmetric vector potential, also in the $\hat{e_\phi}$ direction in spherical co-ordinates.

   (b) i. Show that for a charge density $\rho(\vec{r})$ moving rigidly with a velocity $\vec{v}$ produces a magnetic field given by $\vec{B} = \frac{\vec{v}}{c^2} \times \vec{E}$, where $\vec{E}$ is the electric field produced by imagining $\rho(\vec{r})$ being at rest.

   ii. Using the result of the previous part, derive the results for the magnetic fields produced by

   - a uniform line current I of infinite extent.
   - a uniform surface current on a plane of infinite extent.

   from the corresponding electrostatic problem.

   10 + 5 + (2 + 3) = 20 points

5. A parallel plate capacitor is built out of two conducting circular plates of radius a and interplate distance d. Using long resistance-free wires, the plates are being charged in a way such that the voltage difference across the plates has the time-dependence $\phi = \phi_0 \cos(\omega t)$. Assume the situation $d \ll a \ll \frac{c}{\omega}$. The first inequality allows us to ignore fringing effects at the edge of the plates, while the second inequality allows us to ignore retardation effects, effectively rendering the problem quasi-static.

   (a) Find the electric and magnetic fields everywhere in the region bounded by the two plates at $z = \pm\frac{d}{2}$ and $r < a$.

   (b) What current flows through the wires as a function of time? What is the current density in the top plate as a function of time?

   (c) Find the magnetic field just above the top plate. Show that the magnetic field discontinuity and the current density on the top plate satisfies the matching conditions.

(4 + 4) + (3 + 3) + 6 = 20 points

6. A long, solid cylinder of radius a is permanently polarized, and the polarization is given by $\vec{P} = \frac{P_0}{2}\rho\hat{e_\rho}$, ρ being the radial cylindrical co-ordinate measured from its axis. The cylinder is then set spinning about its axis with an angular velocity ω. Assume that the spinning does not affect the polarization.

   (a) Find the charge density (from all sources) in the cylinder.

   (b) Once steady state is reached, find the magnetic field **on the axis of the cylinder** far away from its top and bottom ends (i.e., you can essentially assume that the cylinder is infinitely extended along its axis.)

4 + 6 = 10 points