Supplementary Examination : Differential Topology. BMath III

Max. Marks : 100 Time : 3 hours

Answer all questions. You may use theorems/propositions proved in the class after correctly stating them. Any other claim must be accompanied by a proof.

(1) State and prove the preimage theorem. Give an example, with justifications, to illustrate the use of the preimage theorem. [10+5]

(2) Define the term *submersion*. State the local submersion theorem. Show that submersions are open maps. [2+3+5]

(3) Define the notion of a Morse function on a manifold. Find the critical points of the function $f : S^1 \longrightarrow \mathbb{R}$ defined by $f(x, y) = xy$. Is f a Morse function. Justify. [4+6+4]

(4) Show that if X is a compact manifold and $f : X \longrightarrow \mathbb{R}$ a Morse function, then f has finitely many critical points. Justify all steps. [15]

(5) Define the notion of a manifold with boundary. Show that the boundary of a manifold is closed. Is $[0, 1] \times [0, 1]$ a manifold with boundary? Justify. [4+2+8]

(6) Discuss the notion of a map being transversal to a submanifold. Suppose that

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

are smooth maps between manifolds and that $W \subseteq Z$ a submanfold. Assume that $g \pitchfork W$. Show that $(g \circ f) \pitchfork W$ if and only if $f \pitchfork g^{-1}(W)$. [2+10]

(7) Let X, Y be manifolds, $Z \subseteq Y$ a submanifold and $f : X \longrightarrow Y$ a smooth map. Under what conditions is the intersection number $I_2(f, Z)$ defined? Give the complete definition. Let $f : S^1 \longrightarrow S^2$ be the map $f(x, y) = (x, y, 0)$. Compute $I_2(f, Z)$ where Z is the equator of S^2. [10]

(8) Show using intersection numbers that the torus $S^1 \times S^1$ is not diffeomorphic to the sphere S^2. [10]