DIFFERENTIAL TOPOLOGY — FEBRUARY 2024

MID-SEMESTER EXAM — 100 POINTS

Notes.

(a) You may freely use any result proved in class unless you have been asked to prove the same. Use your judgement. All other steps must be justified.

(b) $\mathbb{R}$ = real numbers

(c) There are a total of **105** points in this paper. You will be awarded a maximum of **100** points.

---

1. [20 + 5 = 25 Points] Let $f\colon X \to Y$ be a smooth map of manifolds that is one-to-one on a submanifold $Z \subset X$. Suppose that the derivative map $df_p\colon T_p(X) \to T_{f(p)}(Y)$ is an isomorphism for all $p \in Z$.
   (i) Prove that if Z is compact, then f maps an open neighbourhood of Z in X diffeomorphically onto an open neighbourhood of $f(Z)$ in Y.
   (ii) Give an example where Z is not compact, and the conclusion in (i) fails.

2. [10 Points] Let Z be a submanifold of a manifold X. Prove that either Z is an open subset of X or $\dim(Z) < \dim(X)$.

3. [10 + 5 = 15 Points] Suppose X and Z are submanifolds of Y intersecting transversally with each other. Prove that for any $p \in X \cap Z$, we have $T_p(X \cap Z) = T_p(X) \cap T_p(Z)$ as subspaces of $T_p(Y)$. Give an example to show that equality may not hold for a non-transversal intersection.

4. [15 Points] Let X, Y be manifolds. Prove that if f, g, h are smooth maps from X to Y and if there is a smooth homotopy from f to g and a smooth homotopy from g to h, then there is a smooth homotopy from f to h.
(You may assume the following fact: For any two real numbers $a < b$, there exists a smooth function $\rho\colon \mathbb{R} \to \mathbb{R}$ such that $0 \leq \rho(t) \leq 1$ for all t and that $\rho(t) = 0$ if $t \leq a$ while $\rho(t) = 1$ for $t \geq b$.)

5. [20 + 5 = 25 Points] Let X, Y be manifolds with X compact. Let $Z \subset Y$ be a closed submanifold. Explain why, the class of all smooth maps from X to Y that are transversal to Z, is a stable class. Give an example where stability fails if $Z \subset Y$ is not assumed to be closed.

6. [5 + 10 = 15 Points] Let X, Y be manifolds with $\dim(X) > \dim(Y)$.
   (i) Prove that any smooth map $g\colon Y \to X$ is not surjective.
   (ii) Assuming that $\dim(Y) = 1$, prove that any smooth map $f\colon X \to Y$ is not injective.