Mid-Semester Exam B.Math III Year (Differential Topology) 2015

Attempt all questions. Each question carries 5+5=10 marks. Books and notes maybe consulted. Results proved in class, or propositions (with or without proof) from the class notes maybe used after quoting them. Results from exercises, however, must be proved in full if used.)

1. (i): Consider the torus $T^2 \subset \mathbb{R}^3$ obtained by rotating the circle of unit radius centred at $(2,0)$ in the xy-plane around the z-axis. Consider smooth chart on T^2 arising from the local parametrisation:

$$\begin{aligned} f:(0,2\pi)\times(0,2\pi) &\to \mathbb{R}^3 \\ (s,t) &\mapsto ((2+\cos s)\cos t, (2+\cos s)\sin t, \sin s) \end{aligned}$$

   Determine the tangent space $T_{f(s,t)}(T^2)$ as a 2-dimensional subspace of $\mathbb{R}^3$ (i.e., give a basis for it consisting of two vectors in $\mathbb{R}^3$).

   (ii): Consider the smooth map $g: T^2 \to \mathbb{R}$ defined by $g(x,y,z) = x$. Determine the critical points of g on T^2.

2. (i): Let $f: M \to N$ be a smooth submersion of smooth manifolds. Show that f is an open map.

   (ii): Let f be as above, with M compact and N connected. Show that f is surjective.

3. (i): Let U be the subset of $M(2,\mathbb{R})$ consisting of real (2×2) matrices both of whose eigenvalues are real and distinct. Show that U is an open subset of $M(2,\mathbb{R})$.

   (ii): For U as defined in (i), consider the map:

$$\begin{aligned} f: U\times\mathbb{R} &\to \mathbb{R} \\ (A,x) &\to \det(A - xI) \end{aligned}$$

   Show that $f^{-1}(0)$ is a smooth 4-dimensional manifold.

4. (i): Let M be a smooth n-manifold and $f: M \to \mathbb{R}$ be a smooth map. A point $x \in M$ is said to be a *local maximum* (resp. *local minimum*) of f if there exists an open neighbourhood $V \subset M$ of x such that $f(y) \le f(x)$ (resp. $f(y) \ge f(x)$) for all $y \in V$. Show that if x is a local maximum or local minimum (i.e. *local extremum*) for f, then $Df(x): T_x(M) \to \mathbb{R}$ is the zero map. (*Hint:* First show it for the special case of M an open subset of $\mathbb{R}^n$.)

   (ii): Let $g: \mathbb{R}^n \to \mathbb{R}$ be a smooth map, and $a \in \mathbb{R}$ a regular value of f which is in $\operatorname{Im} g$, so that $M = g^{-1}(a)$ is a smooth manifold of dimension $n-1$. Assume that $0 \notin M$, and consider the smooth function $f(x) := \|x\|$ on M. Show that at a local extremum x of f, the derivative $Dg(x)$ (written as a row vector $\left(\frac{\partial g}{\partial x_1}(x), .., \frac{\partial g}{\partial x_n}(x)\right)$ is a non-zero scalar multiple of x. (Since $(Dg(x))^\perp$ is the tangent space $T_x(M) \subset \mathbb{R}^n$, this shows that at a point x of M of extremal distance from the origin, the position vector x is normal to M).