Back-Paper Exam | ISI Bangalore | June 2023

Differential Topology | 100 Points

Notes.

(a) You may freely use any result proved in class or in the textbook unless you have been asked to prove the same. Use your judgement. All other steps must be justified.

(b) $\mathbb{R}$ = real numbers.

(c) All the manifolds here are assumed to be submanifolds in some Euclidean space $\mathbb{R}^N$.

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1. [20 points] Let $f\colon X \to Y$ be a smooth map that is one-one on a compact submanifold Z of X. Suppose that for all $x \in Z$, the derivative $df_x\colon T_x(X) \to T_{f(x)}(Y)$ is an isomorphism. Prove that f maps an open neighbourhood of Z in X diffeomorphically onto an open neighbourhood of $f(Z)$ in Y.

2. [12 points] Show that there exist no submersions of compact manifolds into Euclidean spaces.

3. [20 points] Let Z be a submanifold of a manifold Y and let X be a compact manifold. Show that the set of all maps $X \to Y$ that are transversal to Z is stable (under homotopies).

4. [28 points] Let X be a submanifold in $\mathbb{R}^N$. Define the tangent bundle of X and the normal bundle of its embedding. Prove that the tangent bundle and the normal bundle are manifolds of dimensions $2\dim(X)$ and N respectively.

5. [20 points] State and prove the Brouwer fixed point theorem. You may assume the classification of compact connected 1-manifolds with boundary.