Semestral Examination : Differential Topology. BMath III

Max. Marks : 50 Time : 3 hours

Answer all questions. You may use theorems/propositions proved in the class after correctly stating them. Any other claim must be accompanied by a proof.

(1) (a) Show by an example that a local diffeomorphism between manifolds need not be a closed map. [4]

(b) Let X be a compact manifold and $n \geq 1$. Is it true that every smooth map $f : X \longrightarrow \mathbb{R}^n$ has a critical point if $n < \dim(X)$? [4]

(c) State the preimage theorem. Let I_n denote the $n \times n$ identity matrix and J the $2n \times 2n$ block matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Let $X = \{A \in M_{2n}(\mathbb{R}) \, : \, A^t J A = J\}$. Show that X is a manifold. Find $\dim(X)$. [2+8]

(2) (a) State Sard's theorem. Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}^3$ be two smooth maps. Given $\varepsilon > 0$, show that there exists $v \in \mathbb{R}^3$ such that $||v|| < \varepsilon$ and [2+6]

$$f(\mathbb{R}) \cap (g(\mathbb{R}) + v) = \emptyset.$$

(b) Define the notion of a Morse function on a manifold. Let $U \subseteq \mathbb{R}^n$ be an open set. Prove that a smooth function $f : U \longrightarrow \mathbb{R}$ is Morse if and only if

$$\det(H)^2 + \sum_i \left(\frac{\partial f}{\partial x_i} \right)^2 > 0$$

on U. [2+4]

(c) Let X be a manifold without boundary. Show that the tangent bundle $T(X)$ is manifold with $\dim(T(X)) = 2 \cdot \dim(X)$. [6]

(3) (a) Let X, Y be manifolds, $Z \subseteq Y$ a submanifold and $f : X \longrightarrow Y$ a smooth map. Under what conditions is the intersection number $I_2(f, Z)$ defined? Give the complete definition. Let $f : S^1 \longrightarrow S^2$ be the map $f(x, y) = (x, y, 0)$. Compute $I_2(f, Z)$ where Z is the equator of S^2. [2+4]

(b) Show using intersection numbers that the torus $S^1 \times S^1$ is not diffeomorphic to the sphere S^2. [6]