End-Semester Exam | ISI Bangalore | November 2017

Differential Topology | 100 Points

Notes.
(a) Justify all your steps. You may use any result proved in class unless you have been asked to prove the same.
(b) $\mathbb{R}$ = real numbers, S^k = the unit sphere in $\mathbb{R}^{k+1}$.

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1. [12 points] Show that the antipodal map $x \to -x$ of $S^{2m+1} \to S^{2m+1}$ is homotopic to the identity map.

2. [12 points] Prove or disprove: There exists a smooth map $f\colon \mathbb{R} \to \mathbb{R}$ whose critical values form a dense subset of $\mathbb{R}$.

3. [12 points] Let $U \subset \mathbb{R}^k$ be an open subset and $f\colon U \to \mathbb{R}$ a smooth function. Prove that for almost all k-tuples $\vec{a} = (a_1, \ldots, a_k) \in \mathbb{R}^k$, the function $f_{\vec{a}} := f + a_1 x_1 + \cdots + a_k x_k$ is a Morse function on U.

4. [25 points] For a k-manifold X in $\mathbb{R}^M$ define its tangent bundle $T(X) \to X$ and the normal bundle $N(X) \to X$. Let B denote the open punctured unit ball in $\mathbb{R}^3$, i.e., $B = \{y \in \mathbb{R}^3 \mid 0 < \|y\| < 1\}$. Prove or disprove the following.
   (i) There exist X, k, M such that $T(X)$ is diffeomorphic to B.
   (ii) There exist X, k, M such that $N(X)$ is diffeomorphic to B.

5. [12 points] Suppose that X is a boundaryless manifold and that $\pi\colon X \to \mathbb{R}$ is a smooth function with regular value 0. Then prove that the subset $\{x \in X \mid \pi(x) \geq 0\}$ is a manifold with boundary, the boundary being $\pi^{-1}0$.

6. [15 points] State and prove the Brouwer Fixed-Point Theorem. (*You may assume the theorem that if X is a compact manifold with boundary, then there is no retraction of X onto its boundary*).

7. [12 points] Let Y be a submanifold of $\mathbb{R}^M$ and let $w \in \mathbb{R}^M$. Suppose there is a point $y_0 \in Y$ such that $d(y, w) \geq d(y_0, w)$ for all $y \in Y$. Prove that $w - y_0$ is in the normal space of Y at y_0.