Final Exam ISI Bangalore November 2016

Differential Topology 100 Points

Notes.
(a) Justify all your steps. You may use any result proved in class unless you have been asked to prove the same.
(b) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers,

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1. [28 points] Consider $X = M(n)$, the set of all real $n \times n$ matrices, as a manifold via its natural identification with the linear space $\mathbb{R}^{n^2}$. At any point p, we may also identify the tangent space T_pX with $\mathbb{R}^{n^2}$ or $M(n)$. Let $f\colon X \to \mathbb{R}$ be the determinant function.
   (i) Verify that at the point I corresponding to the identity matrix, the derivative $d_I f\colon M(n) \to \mathbb{R}$ sends any matrix to its trace.
   (ii) Verify that $(d_A f)(A) = n\det(A)$. Deduce that every nonzero $a \in \mathbb{R}$ is a regular value for f.
   (iii) Deduce that $SL(n)$, the set of all matrices in $M(n)$ with determinant 1, is a submanifold of dimension $n^2 - 1$, whose tangent space at I is the set of all matrices with trace 0.

2. [16 points]
   (i) Let C be the plane curve $y = x^2$. Determine which of the following curves intersects transversally with C in $\mathbb{R}^2$. Give brief explanations.

   (a) $y = -x^2$ (b) $x = 0$ (c) $y = -1$

   (ii) If we view each curve above as embedded in $\mathbb{R}^3$ via the canonical embedding of $\mathbb{R}^2$ in $\mathbb{R}^3$, which of the above intersections is transversal ?

3. [16 points]
   (i) For the following functions $f(x, y)\colon \mathbb{R}^2 \to \mathbb{R}$, determine whether the origin is a critical point and if so whether it is nondegenerate:

   (a) $x^2 - 2xy + y^2 + y^3$ (b) $\sin x - \cos^4 y$

   (ii) Prove that the critical non-degenerate points of a smooth function $f : U \to \mathbb{R}$ (where U is an open subset of $\mathbb{R}^n$) are isolated. Deduce the same for a smooth function on an arbitrary manifold.

4. [10 points] If A and B are disjoint, smooth, closed subsets of a manifold X, prove that there is a smooth function ϕ on X such that $0 \le \phi \le 1$ with $\phi = 0$ on A and $\phi = 1$ on B.

5. [10 points] Do any one of the following.
   (i) Prove that the square $S = [0, 1] \times [0, 1]$ in $\mathbb{R}^2$ is not a manifold with boundary.
   (ii) Prove that the locus of $y \ge x^2$ is diffeomorphic to the upper half-plane H^2 in $\mathbb{R}^2$.

6. [10 points] Using the classification of compact connected one-manifolds with boundary, prove that if X is a manifold with boundary ∂X, then there is no smooth map $f\colon X \to \partial X$ whose restriction to the boundary ∂X is identity.

7. [10 points] For a smooth hypersurface $Y \subset \mathbb{R}^M$, show that the normal bundle $N(Y)$ is diffeomorphic to $Y \times \mathbb{R}$.