Semestral Exam B.Math III Year (Differential Topology) 2015

Attempt all questions. Books and notes maybe consulted. Results proved in class, or propositions (with or without proof) from the class notes, or Guillemin and Pollack's book, maybe used after quoting them. Results from exercises in the notes or book, which haven't been solved in class must be proved in full if used.)

1. (i): Prove that the set:

$$X = \{A \in M(2, \mathbb{R}) : \ A \text{ is positive definite with } \det A = 1\}$$

is a smooth 2-dimensional manifold. (Recall A is *positive definite* if $A = A^t$ and $\langle Ax, x\rangle > 0$ for all $x \neq 0$). (8 marks)

(ii): Let $f : X \to Y$ be a smooth map of smooth manifolds with $\dim X = \dim Y$. Suppose f is injective and a local diffeomorphism. Show that $f(X)$ is a smooth manifold and $f : X \to f(X)$ is a diffeomorphism. (7 marks)

2. (i): Consider the map:

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}^3 \\ t &\mapsto (\cos t, \sin t, t) \end{aligned}$$

Let $Z \subset \mathbb{R}^2$ be the unit sphere S^2 centred at the origin. Is f transverse to Z? Justify your answer. (8 marks)

(ii): Let TX be the tangent bundle of a smooth manifold X, and $p : TX \to X$ the bundle projection. Let s be a *smooth vector field* on X (i.e. a smooth map $s : X \to TX$ such that $p \circ s = \mathrm{Id}_X$). Show that s is an embedding of X into TX. (7 marks)

3. (i): Prove that the function:

$$\begin{aligned} f : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z^3 + 6 \sin |z|^2 + 1 \end{aligned}$$

has a zero within the open disc of radius 2 around the origin. (8 marks)

(ii): Let $p_1, p_2, .., p_n$ be real homogeneous polynomials in $(n+1)$ variables, and all of odd degree. Show that there exists a line L through the origin in $\mathbb{R}^{n+1}$ such that $p_{i|L} \equiv 0$ for all i.
(7 marks)

4. (i): Let $Y = S^1 \times S^1$, the 2-torus. Fix a point $p \in S^1$, and define the submanifolds $Z_1 := S^1 \times \{p\}$, $Z_2 := \{p\} \times S^1$ and $Z_3 := (p, p)$. Let X be a compact 2-manifold and $f : X \to Y$ be a smooth map. Show that f is homotopic to a smooth map g such that g is transverse to Z_1, Z_2 and Z_3. (*Hint:* Use the Extension Theorem). (8 marks)

(ii): Show that if $f : S^2 \to S^1 \times S^1$ is any smooth map, then $\deg_2 f = 0$. Prove that S^2 and $S^1 \times S^1$ are not diffeomorphic. (7 marks)