BMath-III, DG2 Mid-semestral exam

Instructions: Total time 3 Hours. Solve as many problems as you like, for a max score of 30.

1. Let M be a parameterized hypersurface in $\mathbb{R}^n$, N its unit normal field and let X, Y be tangential vector fields on M. Prove that
$$\langle \partial_{X(p)}, N(p) \rangle = \langle \partial_{Y(p)}, N(p) \rangle, \text{for all } p \in M.$$
Here, for a smooth vector field Z on M, $\partial_v Z$ denotes the directional derivative of Z in the direction of $v \in T_p(M)$. (8)

2. Give an example of a parameterized hypersurface M in $\mathbb{R}^n$, $n \geq 3$ and a tangential vector field X on M such that for some $p \in M$ and $v \in T_p M$, $\partial_v X$ is not tangential. (4)

3. Let $\alpha : [0, \pi] \longrightarrow S^2$ be the half great circle in S^2 that joins the north pole $P = (0,0,1)$ and the south pole $Q = (0,0,-1)$, defined by $\alpha(t) = (\sin t, 0, \cos t)$. Show that for
$$v = (P, (v_1, v_2, 0)) \in T_P S^2, \ P_\alpha(v) = (Q, (-v_1, v_2, 0)).$$
(8)

4. Prove that in a parameterized hyperplane in $\mathbb{R}^n$, geodesics are straight lines. (4)

5. Let Π in $\mathbb{R}^n$ be a parameterized hyperplane, $p, q \in \Pi$, and $\alpha : [0,1] \longrightarrow \mathbb{R}^n$ a smooth curve in Π such that $\alpha(0) = p$ and $\alpha(1) = q$. Let $(p, v) \in T_p(\Pi)$. Determine $P_\alpha(v)$ and prove that parallel transport in Π is independent of choice of α. (8)

6. Let M be a parameterized hypersurface in $\mathbb{R}^n$ and N be its unit normal. For X, Y tangential vector fields on M, define $[X,Y](p) := \partial_{X(p)} Y - \partial_{Y(p)} X$ and $(D_X Y)(p) = D_{X(p)} Y := \partial_{X(p)} Y - \langle \partial_{X(p)} Y, N(p) \rangle N(p)$ for $p \in M$. Prove that $D_X Y - D_Y X = [X, Y]$. (8)

7. Compute the Weingarten matrix for a parameterized hyperplane in $\mathbb{R}^n$. (6)

8. Let X be the constant vector field on $\mathbb{R}^3$ with $X(p) = (a, b, c)$ for all $p \in \mathbb{R}^3$ and let Y be the vector field given by $Y((x,y,z))) = (xy^2 + 4z, y^2 - x, x + z^3)$. Compute the derivative $\partial_X Y$. Recall that for any point p, $(\partial_X Y)(p) := \partial_{X(p)} Y$. Here we identify the standard basis of $\mathbb{R}^3$ with the basis of the tangent space at any point. (6)

9. Let S be the cylinder in $\mathbb{R}^3$ given by $\sigma(u,v) = (\cos u, \sin u, v)$. Compute the fundamental forms as well as the Gaussian curvature of S. (8)

10. Prove that for the atlas for $\mathbb{R}P^n$ defined in the class, the transition maps are indeed smooth. (6)