BMath-Differential Geometry-II, Back paper Exam

INSTRUCTIONS: All problems carry equal weight and are compulsory. Total time 3 hours. Please use notations and terminology as in the course, use results done in the class without proving them. If you use a problem from some assignment/homework, please provide its solution.

1. Let M be a smooth manifold and $f, g \in C^\infty(M)$. Prove that $\mathrm{d}(fg) = g\mathrm{d}(f) + f\mathrm{d}(g)$. (10)

2. Compute $\mathrm{d}(f^*\omega)$ where $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (-x, x)$ and $\omega = \mathrm{d}y - \mathrm{d}x$. (10)

3. Compute the Lie algebra of the Lie group S^3, realized as the group of norm 1 quaternions. Prove that S^3 is orientable. (5+5)

4. Prove that the holonomy group of the connection on S^2 inherited from the Riemannian connection on $\mathbb{R}^3$ is isomorphic to $\mathrm{SO}(2, \mathbb{R})$ at any point. (10)

5. **(i)** Let M be a smooth manifold admiting an atlas consisting of exactly two charts $\{(U, x),\ (V, y)\}$ with $U \cap V$ (nonempty) connected. Prove that M is orientable. Deduce that S^n is orientable for $n \geq 2$.
   (ii) Prove that S^1 is orientable. (4+1+5)