Differential Geometry-II Final Test (BMath-2023)

Instructions: Total time 3 Hours. Solve problems (even partially), for a maximum score of 50. Use terminologies, notations and results as covered in the course, no need to prove such results. If you wish to use a problem given in an assignment or homework, please supply its full solution.

1. Let M be a smooth connected manifold and D a connection on M. For $P \in M$, let G_P denote the holonomy group of D at P. Prove that for any point $Q \in M$, G_Q is isomorphic to G_P. (15)

2. Let M be a compact smooth manifold of dimension n and $f : M \to \mathbb{R}^n$ a smooth map. Prove that there exists $p \in M$ such that the derivative $Df(p)$ is not injective. (15)

3. Compute the Lie algebra of $\mathrm{SO}(n, \mathbb{R})$ and hence find $\mathrm{Dim}(\mathrm{SO}(n, \mathbb{R}))$. (8+2)

4. Let M, N be smooth manfolds and $f : M \to N$ be a smooth map. Let $g \in C^\infty(N)$. Prove that $\mathrm{d}(g \circ f) = f^*(\mathrm{d}g)$, where $f^*(\omega)$ denotes the pullback of a differential form ω on N, by f. (10)

5. Let $\mathbb{H}$ be the algebra of real quaternions. Identify S^3 with the group of quaternions having norm 1. Identify S^1 as a subgroup of S^3, as the group of norm-1 complex numbers in the usual way. Let $q \in S^3$ and $\phi_q : S^1 \to S^3$ be the multiplication by q, i.e., $\phi_q(z) = qz$. Prove that ϕ_q is an immersion. (10)

6. Prove that any Lie group G is orientable, i.e. there exists a nowhere vanishing top degree differential form on G. (10)

7. Let $\gamma : \mathbb{R} \to S^2$ be the equator $\gamma(t) = (\cos t, \sin t, 0)$, where $S^2 \subset \mathbb{R}^3$ is the 2-sphere with centre as the origin, with Riemannian metric as the induced Riemannian metric from $\mathbb{R}^3$. Let $X(t) = e_3 := (0, 0, 1)$ for all $t \in \mathbb{R}$. Is X parallel along γ? Justify your answer. (10)

8. Let M be a Riemannian manifold and let $d : M \times M \to \mathbb{R}$ denote the induced distance function (metric!) on M. Give an example to show that for $P, Q \in M$, there may not be any curve joining P to Q with length equal to $d(P, Q)$. (5)