DIFFERENTIAL GEOMETRY I MIDTERM EXAMINATION

Total marks: 50
Attempt all questions
Time: 3 hours (10 am - 1 pm)

(1) Consider the parametrized curve $\alpha(s) = (acos(s/c), asin(s/c), bs/c)$ for $s \in \mathbb{R}$, where $c^2 = a^2 + b^2$. Show that the parameter s is the arc length. Determine the curvature and torsion of α. Determine the osculating plane of α. Show that the lines containing $n(s)$ and passing through $\alpha(s)$ meet the z-axis under a constant angle equal to $\pi/2$. Show that the tangent lines to α make a constant angle with the z axis. $(2 + 4 + 2 + 3 + 3 = 14$ marks)

(2) Consider the sphere S^2 in $\mathbb{R}^3$ defined by the equation $x^2 + y^2 + (z - 1)^2 = 1$. Let $N = (0, 0, 2)$ be the north pole, and define $\pi : S^2 - \{N\} \to \mathbb{R}^2$ (called the stereographic projection from the north pole) to be the map which takes any point (x, y, z) of $S^2 - \{N\}$ to the intersection of the xy-plane with the straight line joining N to p. Let $(u, v) = \pi(x, y, z)$, where $(x, y, z) \in S^2 - \{N\}$, and $(u, v) \in xy$-plane (we write a point $(u, v, 0)$ in the xy-plane as (u, v)). Show that the inverse map $\pi^{-1} : \mathbb{R}^2 \to S^2 - \{N\}$ is given by $x = 4u/(u^2 + v^2 + 4)$, $y = 4v/(u^2 + v^2 + 4)$, $z = 2(u^2 + v^2)/(u^2 + v^2 + 4)$. Show that this is a parametrization of $S^2 - \{N\}$. Compute the first fundamental form of the sphere in the above parametrization. (5+6+5 = 16 marks)

(3) Show that the usual notion of a differentiable function $f : S \to \mathbb{R}$ on a regular surface S in $\mathbb{R}^3$ (which is defined using parametrizations) is equivalent to the following: for every $p \in S$, f is the restriction of a differentiable function defined on an open set in $\mathbb{R}^3$ containing p. (10 marks)

(4) Show that if all normals to a connected regular surface pass through a fixed point, then the surface is contained in a sphere. (10 marks)

Date: 9th September, 2019.