Indian Statistical Institute
Mid-Semestral Examination
Differential Geometry - BMath III

Max Marks: 40 Time: 180 minutes.

Give proper justification(s) for your answers.

(1) Define the notion of an integral curve of a vector field.
 (a) Show that the maximal integral curve of the vector field $\mathbb{X}$ on $\mathbb{R}^2$ defined by
$$\mathbb{X}(x_1, x_2) = (x_1, x_2, 1 + x_1^2, 0)$$
 is not defined on the whole of $\mathbb{R}$.
 (b) Let $a, b, c \in \mathbb{R}$ be such that $ac - b^2 > 0$. Show that the maximum and minimum values of the function $f(x_1, x_2) = x_1^2 + x_2^2$ on the ellipse $ax_1^2 + 2bx_1x_2 + cx_2^2 = 1$ are of the form $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ where λ_1, λ_2 are the eigenvalues of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. [2+4+4]

(2) Define the terms : orientation of a n-surface, Gauss map of a n-surface, spherical image of a n-surface.
 (a) Show that any connected n-surface has exactly two orientations.
 (b) Given a subset $A = \{a_1, a_2, a_3\} \subseteq S^2$, construct a 2-surface S in $\mathbb{R}^3$ whose spherical image is A. [2+5+7]

(3) Define the notion of a geodesic in a n-surface S.
 (a) Show that geodesics have constant speed. Give an example to show that a constant speed parametrized curve need not be a geodesic.
 (b) Let S denote the cylinder $x_1^2 + x_2^2 = 1$ in $\mathbb{R}^3$. Show that α is a geodesic of S if and only if α is of the form
$$\alpha(t) = (\cos(at + b), \sin(at + b), ct + d)$$
 for some $a, b, c, d \in \mathbb{R}$. [2+4+10]