B. Math. III – Mid-Term Examination

Introduction to Differential Geometry

September 11, 2009

1. (*Euler's theorem for homogeneous functions*) Let S be an open subset of $\mathbb{R}^n$, and let f be a real valued function defined on S such that $f(\lambda x) = \lambda^p f(x)$ for every real λ and for all x in S for which $\lambda x \in S$. If f is differentiable at x, show that

$$x \cdot \nabla f(x) = pf(x).$$

(Hint: look at $g(\lambda) = f(\lambda x)$)

2. Define the vector product of two vectors in $\mathbb{R}^3$. Prove that the norm of the vector product of the two vectors u and v is the area of the parallelogram generated by them.

3. (*Viviani's curve*) Show that $\gamma(t) = (\cos^2 t - \frac{1}{2}, \sin t \cos t, \sin t)$ is a parametrisation of the curve of intersection of circular cylinder of radius $\frac{1}{2}$ and axis the z-axis with sphere of radius 1 and center $(-\frac{1}{2}, 0, 0)$.

4. Compute the torsion τ and curvature κ of the Viviani's curve given above and verify that:

$$\frac{\tau}{\kappa} = \frac{d}{ds}\left(\frac{\dot{\kappa}}{\tau\kappa^2}\right).$$

5. Show that the ellipse

$$\gamma(t) = (a\cos(t), b\sin(t)),$$

where a and b are positive constants, is a simple closed curve and compute the area of its interior.

6. Find the equation of the tangent plane of the following surface patches at the indicated points:

(1) $\sigma(u, v) = (u, v, u^2 - v^2)$ at $(1, 1, 0)$.
(2) $\sigma(r, \theta) = (r\cosh\theta, r\sinh\theta, r^2)$ at $(1, 0, 1)$.