Each question carries 8 Marks. Answer any five questions. Time 2h30m.

1. (a) If T is $\begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$, determine whether T is parabolic or not. If so, find u and v in $\mathbb{R}^2$ such that $T(u) = u$ and $T(v) = u + v$.

   (b) Suppose A is a parabolic 2×2-matrix. Then show that there is a subspace V of $\mathbb{R}^2$ such that $T(v) = v$ for all $v \in V$ and for $w \in \mathbb{R}^2 \setminus V$, the orbits $(T^n(w))_{n>0}$ grow polynomially.

2. Let A be given by $\begin{pmatrix} 0 & 1 \\ .5 & 1 \end{pmatrix}$.

   (a) Find B so that $B^{-1}AB$ is a diagonal matrix.

   (b) Let (x_n) be a sequence defined by $(x_n, y_n) = A^n(x_{n-1}, y_{n-1})$ with $x_1 = y_1 = 1$. Then find x_n explicitly and find $\lim \frac{x_{n+1}}{x_n}$ and justify your answer.

3. Let A be a 2×2-matrix. Define $||A|| = \sup_{||v||=1} ||Av||$ where $||w||$ is the Euclidean norm for any vector $w \in \mathbb{R}^2$. Then show that

   (a) if $|\det(A)| = 1$, then $||A|| \geq 1$ and

   (b) in general, $||A|| \geq \sqrt{|\det(A)|}$.

4. Let α be a real number and let R_α be the rotation on the circle S^1 by an angle α. For any $A \subset S^1$, $x \in S^1$ and $n \geq 1$, define $F_A(x, n) = |\{k \mid 0 \leq k < n, \;\; R_\alpha^k(x) \in A\}|$. Then

   (a) Show that R_α is an isometry on S^1 for some metric equivalent to the Euclidean metric on S^1.

   (b) If A_1 and A_2 are subsets of S^1, show that $F_{A_1 \cup A_2}(x, n) \leq F_{A_1}(x, n) + F_{A_2}(x, n)$ for all $x \in S^1$ and $n \geq 1$ and the equality occurs if A_1 and A_2 are disjoint.

   (c) Show that $F_A(x, n) \to \infty$ as $n \to \infty$ for all $x \in S^1$ if the interior of A is non-empty and α is irrational.

5. Let $f \colon S^1 \to S^1$ be an orientation-preserving homeomorphism and F be a lift of f. Then show that

   (a) $F(x + m) = F(x) + m$ for all $x \in \mathbb{R}^1$ and all $m \in \mathbb{Z}$,

   (b) F is an increasing map and

   (c) F is invertible and F^{-1} is a lift of f^{-1}.

6. (a) Suppose $F\colon \mathbb{R}^1 \to \mathbb{R}^1$ is $F(x) = 3x + \sin(\frac{x\pi}{2})$ for all $x \in \mathbb{R}^1$. Then determine whether F is a lift of a continuous map of the circle and justify your answer.

   (b) Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism. Then show that the rotation number of f is a root of unity if and only if f has a periodic orbit.

7. Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism whose rotation number $\rho(f)$ is a root of unity. Then show that all periodic orbits have the same period and find the ordering of any periodic orbit.