# DIFFERENTIAL GEOMETRY I BACKPAPER EXAMINATION

Maximum marks: 90
Attempt any SIX questions, each question carries 15 marks.
Time: 3 hours

(1) The tractrix is defined to be the curve $\alpha : (0, \pi/2) \to \mathbb{R}^2$ be given by $\alpha(t) = (sin(t), cos(t) + log(tan(t/2)))$. Show that the length of the segment of the tangent line between the point of tangency (for any point of the tractrix) and the $y$-axis is constantly equal to 1.

(2) Show that the set $\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 - z^2 = 0\}$ is not a regular surface.

(3) Consider the surface of revolution $S$ got by rotating the tractrix (in problem 1) in the $x, z$ plane around the $z$-axis, and compute its Gaussian curvature.

(4) Show that if a curve $C$ in a regular surface $S$ is both a line of curvature and a geodesic, then $C$ is a plane curve.

(5) State the Theorema Egregium of Gauss. Prove that the three surfaces - the sphere, the cylinder and the saddle (defined by $z = x^2 - y^2$) - are not pairwise locally isometric.

(6) Consider two meridians $C_1$ and $C_2$ of a sphere which make an angle $\phi$ at a point $p$. Take the parallel transport of the tangent vector $v$ of $C_1$ along $C_1$ and $C_2$ from the initial point $p$ to the point $q$ where the two meridians meet again, obtaining respectively $w_1$ and $w_2$. Compute the angle between $w_1$ and $w_2$.

(7) Compute the Gaussian curvature of the points of the torus covered by the parametrization $\phi(u, v) = ((a + rcos(u))cos(v), (a + rcos(u))sin(v), rsin(u))$, with $0 < u, v < 2\pi$. Describe the elliptic, hyperbolic and parabolic points on the above parametrized surface.

(8) Define the notion of an umbilical point on a regular surface. If all points on a connected regular surface $S$ are umbilical, then $S$ is either contained in a sphere or a plane.

(9) Define the coefficients of the first and second fundamental form and the Christoffel symbols associated to a parametrization of a regular surface. Write down the equation of Gauss (relating the Christoffel symbols to the Gaussian curvature).

(10) State the local and global Gauss-Bonnet theorems. Prove that if $S$ is a compact connected oriented regular surface of positive Gaussian curvature, then $S$ is homeomorphic to the sphere.