B. Math. III – Back Paper Examination

Introduction to Differential Geometry

January 2015

Instructions: All questions carry equal marks. The 'dot' on top of a function denotes its derivative.

1. Let $\gamma(t)$ be a unit speed curve in $\mathbb{R}^3$. Define the tangent, principal normal and binormal at a point on its trajectory. Derive the Frenet-Serret equations for γ, clearly explaining all the terms involved in the formulae.

2. (Wirtinger's Inequality) Let $F : [0, \pi] \to \mathbb{R}$ be a smooth function with $F(0) = F(\pi) = 0$. Prove that

$$\int_o^\pi \dot{F}^2 dt \geq \int_0^\pi F^2 dt$$

and equality holds if and only if there exists a real number A such that $F(t) = A\sin(t)$ for all $t \in [0, \pi]$.

3. Prove that the sphere of radius $r > 0$ around a point $v \in \mathbb{R}^3$ is a regular surface.

4. Define normal curvature of a curve lying on a regular surface S in $\mathbb{R}^3$. Prove that all curves lying on S and having the same tangent line at a point of S have the same normal curvature at that point.

5. Define Christoffel symbols of a parametric patch of a regular surface S in $\mathbb{R}^3$. Prove that they are invariant under local isometries.

6. Define covariant differentiation of a vector field on a regular patch U of a smooth.surface S. Prove that if S is a plane in $\mathbb{R}^3$, then covariant diffenrentiation agrees with the usual directional derivative in $\mathbb{R}^2$.