DIFFERENTIAL GEOMETRY FINAL EXAM

This exam is of **50 marks** and is **3 hours long** - from 10 am to 1pm. Please **read all the questions carefully**. Please feel free to use whatever theorems you have learned in class after stating them clearly.

I have not used any unfair or illegal means to answer any of the questions in this exam.

Name: **Signature:**

1. Recall that a **generalised helix** is a space curve such that the curvature $\kappa \neq 0$ and all of its tangent vectors make a constant angle with a fixed direction $\mathbf{A}$. Let α be a generalised helix and β the curve optained by projecting α on to a plane orthogonal to $\mathbf{A}$. Let κ_*, τ_* be the curvature and torsion of the curves $*$.

- Prove that $\tau_\alpha/\kappa_\alpha$ is constant. (3)
- Prove that the principle normals to α and β are parallel. (3)
- Calculate κ_β in terms of κ_α. (3)

2. Recall that a curve α is an **asymptotic curve** if $II_P((T(P), T(P)) = 0$ where $T(P)$ is the tangent direction at P. Prove or give a counterexample to the following statements:

- A curve is both planar and an asymptotic curve if and only if it is a line. (3)
- A curve which is both an asymptotic curve and a line of curvature must be planar. (3)
- A curve is both a geodesic and an asymptotic curve if and only if it is a line. (3)
- If a curve is both a geodesic and a line of curvature then it must be planar. (3)

3. Let $\mathbf{M}$ be the surface given by the parametrization

$$x(u, v) = a(\cosh(u)\cos(v), \cosh(u)\sin(v), u)$$

, $-1 \leq u \leq 1$, $0 \leq v \leq 2\pi$. Let $\partial(\mathbf{M})$ denote the boundary circles $u = \pm 1$.
Compute:

- The First Fundamental form I_P. (2)
- The Second Fundamental form II_P. (2)
- The matrix of the shape operator S_P. (2)
- The Christoffel symbols Γ^*_{**}. (6)
- The Mean curvature H. (2)
- The Gaussian curvature K. (2)
- The surface area of $\mathbf{M}$. (3)
- The integral $\int_{\partial(\mathbf{M})} \kappa_g ds$ where κ_g is the geodesic curvature of the boundary circles. (4)
- The Total curvature $\iint_{\mathbf{M}} K dA$ (3)
- The Euler characteristic $\chi(\mathbf{M})$. (3)

Some useful formulae

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \text{ and } II = \begin{pmatrix} \ell & m \\ m & n \end{pmatrix}$$

$$\begin{pmatrix} \Gamma^u_{uu} \\ \Gamma^v_{uu} \end{pmatrix} = I^{-1} \begin{pmatrix} \frac{1}{2}E_u \\ F_u - \frac{1}{2}E_v \end{pmatrix}, \qquad \begin{pmatrix} \Gamma^u_{uv} \\ \Gamma^v_{uv} \end{pmatrix} = I^{-1} \begin{pmatrix} \frac{1}{2}E_v \\ \frac{1}{2}G_u \end{pmatrix}, \qquad \begin{pmatrix} \Gamma^u_{vv} \\ \Gamma^v_{vv} \end{pmatrix} = I^{-1} \begin{pmatrix} F_v - \frac{1}{2}G_u \\ \frac{1}{2}G_v \end{pmatrix}$$

$$\ell_v - m_u = \ell \Gamma uv^u + m(\Gamma^v_{uv} - \Gamma^u_{uu}) - n\Gamma^v_{uu}$$

$$m_v - n_u = \ell \Gamma vv^u + m(\Gamma^v_{vv} - \Gamma^u_{uv}) - n\Gamma^v_{uv}$$