FINAL EXAM ISI BANGALORE DECEMBER 2021

DIFFERENTIAL GEOMETRY 100 POINTS

Notes.
(a) You may freely use any result proved in class or in the textbook. All other steps must be justified.
(b) $\mathbb{R}$ = real numbers.

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1. [16 points] For the curve $\gamma(t) = (t,\, \cos(\pi t),\, e^t)$, calculate the principal normal $\mathbf{N}$ and the binormal $\mathbf{B}$ at $t = 0$. (Note: If you use any formula not derived in the notes or the text-book, then you must prove it first.)

2. [16 points] Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$ through $p = (0, 0, 0)$. Let H be a plane through p and let $\alpha_H(t)$ be the curve obtained by orthogonally projecting $\gamma(t)$ to H. Prove that H is the normal plane to $\gamma(t)$ at $t = 0$ (i.e., H is spanned by the principal normal and the binormal at $t = 0$) iff $\alpha_H(t)$ is singular at $t = 0$.

3. [16 points] Let $f(t), g(t)$ be smooth functions on $\mathbb{R}$. Prove that the intersection of the two surfaces $xf(x) + yz + 1 = 0$ and $x^2 e^{g(x)} + y^2 - z^4 = 0$ in $\mathbb{R}^3$ is either empty or a regular curve in $\mathbb{R}^3$.

4. [16 points] Prove that $z = |x|$ does not define a smooth surface in $\mathbb{R}^3$.

5. [18 points] Let $\sigma(u, v) = (f(u, v),\, g(u, v),\, h(u, v))\colon \mathbb{R}^2 \to S$ be a parametrised surface in $\mathbb{R}^3$. Consider the parametrised surfaces $\widehat{\sigma}(u, v)\colon \mathbb{R}^2 \to \widehat{S}$ and $\sigma^*(u, v)\colon \mathbb{R}^2 \to S^*$ given by

$$\widehat{\sigma}(u, v) = (f(u, v) + g(u, v),\ f(u, v) - g(u, v), \sqrt{2} \cdot h(u, v)),$$
$$\sigma^*(u, v) = (f(u + v, u - 2v),\ g(u + v, u - 2v),\ h(u + v, u - 2v)).$$

For each pair of surfaces among S, $\widehat{S}$ and S^*, find a diffeomorphism between the two surfaces that is at least one among (isometric/conformal/equi-areal). (Here f, g and h are arbitrary.) Justify your answer.

6. [18 points] Consider the following two surfaces (where $u \in \mathbb{R}$, $0 < v < 2\pi$)

$$\alpha(u, v) = (v,\, \sinh(u)\cos(v),\, \sinh(u)\sin(v)), \qquad \beta(u, v) = (u,\, \cosh(u)\cos(v),\, \cosh(u)\sin(v)).$$

(i) Compute the first and second fundamental forms of these surfaces and determine their principal and Gaussian curvatures as functions of u, v.
(ii) Explain why the two surfaces are isometric.