Indian Statistical Institute
Semestral Examination
Differential Geometry I - BMath III

Max Marks: 60 Time: 180 minutes.

Answer all questions with proper justifications.

(1) (a) Show that the set S of all unit vectors at all points in $\mathbb{R}^2$ is a 3-surface in $\mathbb{R}^4$. [4]
(b) Let $a = (a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1}$, $a \neq 0$. Show that the spherical image of an n-surface S in $\mathbb{R}^{n+1}$ is contained in the n-plane $\sum_{i=1}^{n+1} a_i x_i = 0$ if and only if for every $p \in S$ there is an open interval I about 0 such that $p + ta \in S$ for all $t \in I$. [8]

(2) (a) Let $\alpha : [a, b] \longrightarrow S$ be a unit speed parametrized curve in the oriented 2-surface S in $\mathbb{R}^3$. Define a function $\kappa : [a, b] \longrightarrow \mathbb{R}$ by

$$\kappa = (\dot{\alpha})' \cdot (\mathbb{N} \circ \alpha \times \dot{\alpha})$$

where $\mathbb{N}$ is the orientation of S. Show that $\kappa(t) = 0$ for all t if and only if α is a geodesic. [6]
(b) Show that a parametrized curve α in the unit sphere $x_1^2 + \cdots + x_{n+1}^2 = 1$ is a geodesic if and only if α is of the form

$$\alpha(t) = (\cos at)e_1 + (\sin at)e_2$$

for some orthonormal pair of vectors $\{e_1, e_2\}$ in $\mathbb{R}^{n+1}$ and some $a \in \mathbb{R}$. [8]

(3) (a) Compute the following integrals :
(i) $\int_C -x_2 \, dx_1 + x_1 \, dx_2$, where C is the ellipse $(x_1^2/a^2) + (x_2^2/b^2) = 1$, $a, b \neq 0$.
(ii) $\int_\alpha x_1 \, dx_1 + x_2 \, dx_2$, where α is any curve from $(0, 0)$ to $(1, 1)$. [8]
(b) Let S be the n-surface $S = f^{-1}(r^2)$ $(r > 0)$ oriented by $\nabla f / ||\nabla f||$ where

$$f(x_1, \ldots, x_{n+1}) = x_2^2 + x_3^2 + \cdots + x_{n+1}^2.$$

Compute the normal curvatures $k(v)$ for each tangent direction v, the principal curvatures and principal curvature directions, the Gauss-Kronecker curvature and mean curvature of S at the point $p = (0, 0, \ldots, 0, r)$. [10]

(4) (a) Find the Gaussian curvature of the parametrized 2-surface

$$\varphi(t, \theta) = (t \cos \theta, t \sin \theta, \theta).$$

[10]
(b) Find the area of the parametrized 2-surface φ defined by

$$\varphi(\theta, \phi) = (a \cos \theta, a \sin \theta, b \cos \phi, b \sin \phi)$$

$0 < \theta < 2\pi$, $0 < \phi < 2\pi$. [6]