Semestral Exam B.Math III Year (Differential Geometry) 2015

Attempt all questions. Books and notes maybe consulted. Results proved in class, or propositions (with or without proof) from the class notes maybe used after quoting them. Results from exercises in the notes or Pressley's book, which haven't been solved in class must be proved in full if used.)

1. (i): Let $X = f^{-1}(0) \subset \mathbb{R}^3$ be a level surface of a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ having 0 as a regular value. For some fixed $x \in X$, let $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^3$ be a smooth curve with $\gamma(0) = x$ and $\gamma'(0) = v \neq 0$ such that $v \notin T_x(X)$. Show that the function $g(t) := (f \circ \gamma)(t)$ changes sign at $t = 0$. Conclude that for ϵ small enough, $\gamma(-\epsilon, \epsilon) \cap X = \{x\}$. (7 mks)

   (ii): Let X as above be *compact*, and $L = a + \mathbb{R}v$ be a ray in $\mathbb{R}^3$ with $\|v\| = 1$ such that $L \cap X = \{x_1, .., x_n\}$. Assume that $v \notin T_{x_i}(X)$ for all $1 \leq i \leq n$. Show that n is even.
   (*Hint:* Let S be a large sphere centred at a of radius A, surrounding X and not meeting it. Show that $f(a + Av)$ and $f(a - Av)$ have the same sign.) (8 mks)

2. (i): Show that there does not exist any smooth simple closed plane curve of length 2π with the signed curvature function $k_s(t) = \sin t$ where $t \in [0, 2\pi]$ is the arc length parameter. (7 mks)

   (ii): Consider the torus:
   $$T^2 = \left\{((2 + \cos s)\cos t,\ (2 + \cos s)\sin t,\ \sin s) \in \mathbb{R}^3 :\ s, t \in [0, 2\pi]\right\}$$
   Compute the normal and geodesic curvature of the curve $c : [0, 4\pi] \to T^2$ defined by:
   $$c(t) = \left(2\cos\frac{t}{2}, 2\sin\frac{t}{2}, 1\right)$$
   where t is the arc length parameter. (8 mks)

3. Consider the surface:
   $$X = \{(x, y, z) \in \mathbb{R}^3 : y = x^3\}$$
   (i): Compute its mean curvature at a point (x, y, z). (8 mks)

   (ii): Show that an open subset of X and an open subset of a sphere cannot be isometric. (7 mks)

4. Let $X \subset \mathbb{R}^3$ be a compact surface of genus 2. Let $K(x)$ denote its scalar curvature.

   (i): Show that :
   $$\inf_{x \in X} K(x) = -\lambda$$
   for some $\lambda > 0$ (7 mks)

   (ii): Show that $\text{Area}(X) \geq \frac{4\pi}{\lambda}$. (8 mks)