End-Semester Exam — ISI Bangalore — November 2013

Elementary Differential Geometry — 100 Points

Notes.

(a) Begin each answer on a separate sheet and ensure that the answers to all the parts to a question are arranged contiguously.

(b) Assume only those results that have been proved in class. All other steps should be justified.

(c) $\mathbb{R}$ = real numbers.

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1. [12 points] Let S be a surface, regularly embedded in some $\mathbb{R}^n$. Suppose there exist two charts on S, namely (U_i, σ_i) for $i = 1, 2$, such that U_i are connected and the intersection of their images on S, namely $\sigma(U_1) \cap \sigma(U_2)$, is also connected. Prove that S is orientable.

2. [24 points] In each of the following cases, give an example of a diffeomeorphism ϕ from some open set U in $\mathbb{R}^2$ to an open subset of some surface S regularly embedded in $\mathbb{R}^3$, satisfying the given conditions:

(i) ϕ is an isometry but S is not a plane.

(ii) ϕ is conformal but not an isometry.

(iii) ϕ is equi-areal but not an isometry.

(Note: While specifying ϕ, you must, very briefly, verify that ϕ is one-to-one and regular.)

3. [28 points] Let $\alpha(u) = (f(u), 0, g(u))$ be a unit-speed curve in the XZ-plane in $\mathbb{R}^3$ with $f(u) > 0$ so that $\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$ gives a chart on the surface obtained by revolving $\alpha(u)$ around the Z-axis. Let $\dot{\alpha}$ denote derivative with respect to u, etc.

(i) Verify that the first fundamental form in the uv-plane is $du^2 + f^2 dv^2$.

(ii) Verify that the second fundamental form is $(\dot{f}\ddot{g} - \ddot{f}\dot{g})du^2 + f\dot{g}dv^2$.

(iii) Compute the Gaussian curvature K.

(iv) Prove that $\dot{f}\ddot{f} + \dot{g}\ddot{g} = 0$ and use this to show that the formula for K in (iii) can be simplified to $(-\ddot{f}/f)$.

4. [24 points] For this question you may use the formula in question 3 above even if you haven't proved it.

(i) Prove that there exists a surface in $\mathbb{R}^3$ that has constant Gaussian curvature -1 everywhere.

(ii) Give an example of a surface in $\mathbb{R}^3$ that has constant Gaussian curvature +2 everywhere.

(iii) Give an example of a surface that has points of positive, negative and zero curvature on it. (You must also exhibit one example of each such point.)

5. [12 points] Let X_1, X_2, X_3 denote co-ordinate functions on $\mathbb{R}^3$. Let S be a regularly embedded surface in $\mathbb{R}^3$ and (U, σ) a chart on S. Prove or disprove: For $i = 1, 2, 3$, there always exist C^∞ functions f_i on U such that f_i do not simultaneously vanish on any point of U and such that $\Sigma_i f_i \sigma^*(dX_i) = 0$.