Indian Statistical Institute
B.Math. (Hons.) III Year
First Semester Exam 2006-07
Introduction to Differential Geometry

Time: 3 hrs Date:27-11-06

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Attempt all questions, they carry equal marks. You may use any result proved in the course.

1. (a) Let γ be a regular curve (not necessarily unit speed) on a surface patch σ. Prove that the normal curvature of γ is given by

$$\mathcal{K}_n = \frac{L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2}{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}$$

   where $E\ du^2 + 2F\ dudv + G\ dv^2$ and $L\ du^2 + 2M\ dudv + N\ dv^2$ are the first and second fundamental forms of σ respectively.

   (b) Let γ be a curve on the unit sphere S^2. Compute the normal curvature of γ.

2. Let S be a compact surface in $\mathbb{R}^3$ whose Gaussian curvature K is positive everywhere. Show that S is diffeomorphic to a sphere. Is the converse true?

3. Let σ be a surface patch whose Gaussian curvature $K \leq -1$ everywhere. Let γ be a geodesic n-gon contained in σ. Show that $n \geq 3$ and when $n = 3$, the area enclosed by γ is atmost π.

4. Let S be the ellipsoid

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1, \quad a, b > 0.$$

   Prove that $\iint_S K\ dA = 4\pi$, where K denotes the Gaussian curvature of S.

5. Let a triangulation of a compact surface S in $\mathbb{R}^3$ have V vertices, E edges and F triangles. Let χ be the Euler characteristic of S. Show that

$$\begin{aligned} 3F &= 2E, \quad E = 3(V-\chi) \text{ and} \\ V &\geq \frac{1}{2}(7+\sqrt{49-24\chi}). \end{aligned}$$

6. A triangulation of S^2 has F triangles and r triangles meet at each vertex. Show that $V = \frac{3F}{r}$. Compute E and show that $\frac{6}{r} - \frac{4}{F} = 1$. Here V, E are the number of vertices and edges respectively, in the triangulation.