Indian Statistical Institute

Midterm Examination 2023-2024

B.Math Third Year Complex Analysis 23 February, 2024 Instructor : Jaydeb Sarkar

Time : 3 Hours Maximum Marks : 100

(i) $B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$. (ii) $C_r(z_0) := \partial B_r(z_0)$. (iii) $\text{Hol}(U) = \{f : U \to \mathbb{C} \text{ holomorphic}\}$. (iv) $\mathbb{D} = B_1(0)$.

(1) (10 marks) Prove that if f is a nonconstant entire function, then $f(\mathbb{C})$ is dense in $\mathbb{C}$.

(2) (10 marks) Let $f \in \text{Hol}(\mathbb{D})$. Suppose

$$f\Big(\frac{1}{n}\Big) = 1 \qquad (n \in \mathbb{N}).$$

Compute $f(\frac{2023}{2024})$.

(3) (8+7=15 marks) Evaluate the following integrals:

$$(i) \int_{C_6(0)} \frac{z^9}{z^5 - 5^5} dz. \qquad (ii) \int_{C_1(0)} z|z|\, dz.$$

(4) (15 marks) Let Ω be a domain. If f and zf are both harmonic functions on Ω, then prove that $f \in \text{Hol}(\Omega)$.

(5) (15 marks) Let $f \in \text{Hol}(\mathbb{D})$. True of false (with justification)?

$$\text{“}f^{(n)}(0) = \frac{1}{2\pi i} \int_{C_{\frac{1}{2}}(0)} \frac{f(z)^n}{z}\, dz \qquad (n \geq 1)\text{”}.$$

(6) (15 marks) Let $f \in C(\overline{\mathbb{D}})$ be a nonconstant function, and suppose $f|_{\mathbb{D}} \in \text{Hol}(\mathbb{D})$. If

$$|f(z)| = 1 \qquad (z \in C_1(0)),$$

then prove that $0 \in f(\mathbb{D})$.

(7) (15 marks) Let $U = \{z \in \mathbb{C} : 0 < |z| < 1\}$, and let $f \in \text{Hol}(U)$. If

$$|f(z)| \leq \log(1/|z|) \qquad (z \in U),$$

then prove that $f \equiv 0$.

(8) (15 marks) Let $f : [0, \infty) \to \mathbb{R}$ be a bounded and continuous function. Prove that $F \in \text{Hol}(\{z \in \mathbb{C} : Re(z) > 0\})$, where

$$F(z) = \int_0^\infty f(t) e^{-zt}\, dt.$$