Midterm - Complex Analysis (2022-23)
Time: 2.5 hours.

Attempt all questions, giving proper explanations.
You may quote any result proved in class without proof.

1. Let γ be the boundary of the triangle $\{0 < y < 1 - x,\ 0 < x < 1\}$ with the usual counter-clockwise orientation.
   (a) Evaluate $\int_\gamma \mathrm{Re}(z)\, dz$. **[4 marks]**
   (b) Conclude that $\mathrm{Re}(z)$ is not a holomorphic function. **[1 mark]**

2. Show that
$$\left| \int_{|z-1|=1} \frac{e^z}{z+1} dz \right| \leq 2\pi e^2. \qquad \textbf{[4 marks]}$$

3. Compute
$$\int_{|z|=2} \frac{1}{z^4 - 1} dz,$$
where the integral over the circle is in the positive orientation. **[4 marks]**

4. Say that a twice continuously differentiable real-valued function $u : \mathbf{R}^2 \to \mathbf{R}$ is harmonic if $\Delta u(x, y) = 0$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.
   (a) If f is holomorphic in an open set Ω, then show that the real part of f is harmonic. **[2 marks]**
   (b) Let u be a real-valued function defined on the unit disc $\mathbb{D}$. Suppose that u is twice continuously differentiable and harmonic. Prove that there exists a holomorphic function f on $\mathbb{D}$ such that $\mathrm{Re}(f) = u$. **[4 marks]**
   [HINT: If there is such an f then $f'(z) = 2\partial u/\partial z := \partial u/\partial x - i\partial u/\partial y$. Therefore, let $g(z) = 2\partial u/\partial z$ and prove that g is holomorphic. Why can one find F with $F' = g$? Prove that $\mathrm{Re}(F)$ differs from u by a real constant.]

5. Suppose f is holomorphic in an open set that contains the *closed rectangle* $R = \partial R \cup R^o$, where ∂R denotes the boundary of R, and R^o denotes the interior.
   (a) Let $z \in R^o$, and let $C_\epsilon(z)$ be a circle of radius ϵ centered around z with positive orientation. Show that
$$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{C_\epsilon(z)} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z). \qquad \textbf{[3 marks]}$$
   (b) Show that
$$f(z) = \frac{1}{2\pi i} \int_{\partial R} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{for any } z \in R^o,$$
   where the integral over ∂R is taken in the positive orientation. **[3 marks]**

6. Let f be holomorphic on the punctured disc $D(z_0, R)\backslash z_0$ and let z_0 be a pole for f. Prove that for any $r \in (0, R)$ there is an $m \in (0, \infty)$ such that $f\left(D(z_0, r)\backslash z_0\right) \supset \{z : |z| > m\}$. **[5 marks]**
   (HINT: Use the open mapping theorem to $g(z) = \frac{1}{f(z)}$)