Indian Statistical Institute

Midterm Examination 2019-2020

B.Math Third Year   Complex Analysis   September 06, 2019   Instructor : Jaydeb Sarkar

Time : 3 Hours   Maximum Marks : 100

(i) $B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$. (ii) $C_r(z_0) := \partial B_r(z_0)$. (iii) $\text{Hol}(U) = \{f : U \to \mathbb{C} \text{ holomorphic}\}$. (iv) $\mathbb{D} = B_1(0)$.

(1) (10+10=20 marks) Use Cauchy's integral formula to compute

$$(i) \int_{C_{\sqrt{2}}(2)} \frac{z+1}{z^2(z-1)} dz, \qquad (ii) \int_{C_1(0)} |z+1|^2 dz.$$

(2) (15 marks) Show that $f \in \text{Hol}(\mathbb{D})$ where

$$f(z) = \int_0^1 \frac{1}{1-tz} dt.$$

(3) (15 marks) Prove that if an entire function f is not constant, then $f(\mathbb{C})$ is dense in $\mathbb{C}$.

(4) (15 marks) Let $f \in \text{Hol}(\mathbb{D})$ with power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Prove that

$$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$$

is an entire function.

(5) (15 marks) Let f be an entire function. If

$$|f(z)| \leq \sqrt{1 + |z|} \qquad (z \in \mathbb{C}),$$

then prove that f is a constant function.

(6) (15 marks) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a harmonic function. If

$$u(x, y) \geq 0 \qquad (x, y \in \mathbb{R}),$$

then prove that u is a constant function.

(7) (15 marks) Let $f_1, f_2 \in \text{Hol}(\mathbb{D})$ and assume that $|f_1'(0)| > |f_2'(0)|$. Prove that $f_1 + \bar{f}_2$ is one-to-one in an open neighborhood of the origin.