Indian Statistical Institute

Midterm Examination 2018-2019

B.Math Third Year    Complex Analysis    September 10, 2018    Instructor : Jaydeb Sarkar

Time : 3 Hours    Maximum Marks : 100

---

(i) $\mathcal{D}$ = a domain in $\mathbb{C}$. (ii) $B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$. (iii) $C_r(z_0) := \partial B_r(z_0)$. (iv) $Hol(U) = \{f : U \to \mathbb{C}$ holomorphic $\}$. (v) $Cont(U) = \{f : U \to \mathbb{C}$ continuous $\}$.
(vi) $\mathbb{D} = B_1(0)$.

(1) (10 marks) Let $f \in Hol(\mathbb{D}) \cap Cont(\overline{\mathbb{D}})$. Suppose that $|f(z)|$ is constant for all $z \in \partial\mathbb{D}$. Prove that $f$ must have a zero in $\mathbb{D}$.

(2) (10 marks) Show that if $f$ is a non-constant entire function, then $f(\mathbb{C})$ is dense in $\mathbb{C}$.

(3) (10 marks) Let $\{f_n\}_{n\geq 1} \subseteq Hol(\mathcal{D})$ be a sequence of holomorphic functions which converges uniformly on every compact subset of $\mathcal{D}$. Show that the sequence of derivatives $\{f_n'\}_{n\geq 1}$ also converges uniformly on every compact subset of $\mathcal{D}$.

(4) (10 marks) Compute the radius of converges of the power series $\sum_{n=0}^{\infty} a_n(z+2)^n$ where

$$\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} a_n(z+2)^n,$$

for all $z \in \mathbb{D}$.

(5) (10 marks) Does there exist a function $f \in Hol(\mathbb{D})$ such that

$$f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{n^3},$$

for all $n \geq 1$?

(6) (10 marks) Let $m \in \mathbb{N}$ and $z_0 \in \mathbb{C}$. Suppose that $f, g \in Hol(B_e(z_0))$, $g(z) \neq 0$ and $f(z) = (z - z_0)^m g(z)$ for all $z \in B_e(z_0)$. Compute

$$\frac{1}{2\pi i} \int_{C_1(0)} \frac{f'(z)}{f(z)} dz.$$

(7) (15 marks) Let $u(x, y) = x^2 - y^2 - 2xy - 2x + 3y + 5$, $(x, y) \in \mathbb{R}^2$. Prove that $u$ is a harmonic function and find a harmonic conjugate of $u$.

(8) (15 marks) Let $\{f_n\}_{n\geq 1} \subseteq Hol(\mathbb{D}) \cap Cont(\overline{\mathbb{D}})$. If $\{f_n\}_{n\geq 1}$ converges uniformly to a function $f$ on $\partial\mathbb{D}$, then prove that $f$ can be extended to a function $\tilde{f}$ on $\overline{\mathbb{D}}$ which is analytic on $\mathbb{D}$.

(9) (15 marks) Let $f : \mathbb{D} \to \mathbb{C}$ be a function. If $f^2, f^3 \in Hol(\mathbb{D})$, then prove that $f$ is analytic.