Indian Statistical Institute

Mid-Semestral Examination 2014-2015

B.Math Third Year

Complex Analysis

Time : 3 Hours Date : 08.09.2014 Maximum Marks : 100 Instructor : Jaydeb Sarkar

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(i) Answer all questions. (ii) $B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$. (iii) $C_r(z_0) := \{z \in \mathbb{C} : |z - z_0| = r\}$. (iv) $U \subseteq \mathbb{C}$ open. (v) $\text{Hol}(U) = \{f : U \to \mathbb{C} \text{ holonorphic }\}$.

Q1. (10+10 = 20 marks) True or False (with justification)?

(i) $\sum_{n=0}^{\infty} \frac{2^n}{n!} z^{3n}$ is an entire function.

(ii)

$$\frac{1}{\pi i} \int_{C_e(0)} \frac{1 + ez + e^z}{(z-1)^3}\, dz = e.$$

Q2. (15 marks) Let $f \in \text{Hol}\,(U)$ and f has a zero of order m at $z_0 \in U$. Prove that

$$\frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f'(z)}{f(z)}\, dz = m,$$

for some $r > 0$.

Q3. (15 marks) Prove that $\sum_{n=0}^{\infty} a_n z^n$ is holomorphic on $B_1(0)$, where $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, $a_n \in \mathbb{C}$.

Q4. (8+7 = 15 marks) Let $f \in \text{Hol}\,(\mathbb{C})$. What can you conclude about f:

(i) when $f(\mathbb{C}) \cap B_1(0)$ is an empty set.

(ii) when f, restricted to $\mathbb{R}$, is a 2π-periodic function.

Q5. (15 marks) Let f be a continuous function on U and $e^{f(z)} = z$ for all $z \in U$. Prove that $f \in \text{Hol}(U)$. Compute f'.

Q6. (15 marks) Let D be a domain in $\mathbb{C}$ and $f \in \text{Hol}(D)$. Prove that

$$\int_\gamma \overline{f(z)} f'(z)\, dz$$

is purely imaginary, where $\gamma \subseteq D$ is a simple closed curve. [Hint: 2Real $z = z + \bar{z}$.]

Q7. (15 marks) Let $H := \{z \in \mathbb{C} : \text{Real } z \leq 0\}$. Prove that

$$|e^z - e^w| \leq |z - w|,$$

for all $z, w \in H$.

Q8. (15 marks) Let $f \in \text{Hol}(\mathbb{C})$ and

$$f''(\frac{1}{n}) + f(\frac{1}{n}) = 0,$$

for all $n \in \mathbb{N}$. Characterize f.