Indian Statistical Institute, Bangalore Centre
B.Math (Hons.) III Year, First Semester
Mid-Sem Examination - 2013-2014
Complex Analysis

Time: 3 Hours | September 06, 2013 | Instructor: Bhaskar Bagchi

Maximum Marks : 100.
Direction : Answer any five of the following six Questions.

1. (a) Let $f : X \to \mathbb{C}^*$ be the identity function. If $X = S^1$ then show that f has no continuous branch of its argument. If $X \subset S^1$ then show that it has a continuous branch of its argument.

   (b) Define $Ind\ (\gamma, z)$ for any closed path γ in $\mathbb{C}$ and $z \in \mathbb{C} \backslash tr(\gamma)$. If γ is piecewise smooth then derive the integral formula for this index. (You may assume the Path Lifting Lemma without proof.)

   (10+10=20)

2. (a) Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(x+iy) = \sum_{j,\ k=0}^{n} a_{jk}\ x^j y^k$, where a_{jk} are complex constants and $x, y \in \mathbb{R}$. If f is holomorphic then show that $f(z) = \sum_{j=0}^{n} a_{j0}\ z^j$ for all $z \in \mathbb{C}$.

   (b) Let f be an entire function which maps $\mathbb{R}$ into $\mathbb{R}$. Then show that $f(\bar{z}) = \overline{f(z)}$ for all $z \in \mathbb{C}$.

   (10+10=20)

3. (a) State and prove Guvsat's Lemma.

   (b) When is a planar domain called simply connected? If Ω is a planar domain such that $\int_\gamma f = 0$ for every closed path γ in Ω and every holomorphic function f defined on Ω then show that Ω must be simply connected.

   (15+5=20)

4. (a) Prove that the sine function (originally defined on the real line) can be extended to an entire function.

   (b) Prove that this entire function maps the domain $\{z = x+iy : 0 < x < 2\pi,\ y > 0\}$ bijectively onto the complement in $\mathbb{C}$ of $[-1, 1] \bigcup [0, i\ \infty)$.

   (5+15=20)

5. Let $\{f_n : n \geq 1\}$ and f be complex valued continuous functions on a domain Ω, such that $f_n \to f$ locally uniformly on Ω. If each f_n is holomorphic, then show that
   (a) f is holomorphic,
   (b) $f_n' \to f'$ locally uniformly on Ω.

   (5+15=20)

6. Let f be a holomorphic function on a planar domain Ω. Suppose f is not the identically zero function. Then show that

(a) The zero-set of f is discrete in Ω.

(b) Give an example of such a function f (on some planar domain Ω) such that the zero-set of f has a limit point in the boundary of Ω.

(12+8=20)