Indian Statistical Institute, Bangalore

B.Math (Hons.) III Year, First Semester

Mid-Semester Examination

Complex Analysis

Time: 3 hours September 23, 2011 Instructor: B.Bagchi

Maximum marks: 100

1. For any fixed real number c, let $r_c : \mathbb{R} \to \mathbb{C}$ denote the path given by $r_c(x) = x + ic$.

   a) Use the fundamental theorem to prove that $\int\limits_{r_c} e^{-z^2} dz$ is independent of $c \epsilon \mathbb{R}$.

   b) Hence find a formula for the function

   $\phi : \mathbb{R} \to \mathbb{R}$ given by

   $\phi(t) = \int\limits_{-\infty}^{\infty} e^{itx}\ e^{-x^2}\ dx$

   (You may use without proof the fact that $\int\limits_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}$).

   [10+5 =15]

2. Let f be an entire function, i.e f is holomorphic on the entire complex plane.

   a) Show that there is a power series around 0 representing f throughout $\mathbb{C}$.

   b) Say $f(z) = \sum\limits_{n=0}^{\infty} a_n\ z^n, z \ \epsilon\ \mathbb{C}$. Then it is easy to see that for any $r > 0$, $\frac{1}{2\pi} \int\limits_{0}^{2\pi} \mid f(re^{i\theta}) \mid^2 d\theta = \sum\limits_{n=0}^{\infty} \mid a_n \mid^2\ r^{2n}$.

   Now suppose there is a non-negative integer m and a constant $c > 0$ such that $\mid f(z) \mid \leq c \mid z \mid^m$ for all z in $\mathbb{C}$.

   Then use the above formula to show that f must be a complex polynomial of degree $\leq m$.

   [5+20 =25]

3. Let f be an entire function given by the formula

$$f(x+iy) = \sum_{k,l=0}^{N} a_{kl}\ x^k\ y^l,\ \ x, y \epsilon \mathbb{R},$$

   where a_{kl} are complex constants.

   Then show that f must be a complex polynomial.

   (Hint: You may use the results from question 2.) [15]

4. Let $f : \Omega \to \mathbb{C}$ be a continuous function such that $\int_r f = 0$ for all closed paths r in Ω. Then show that f is holomorphic. [20]

5. Let Ω be a convex domain and $f : \Omega \to \mathbb{C}\backslash\{0\}$ be holomorphic. Then show that there is a holomorphic branch of the logarithm of f.

   (Hint: First find a primitive of such a branch.) [25]