Indian Statistical Institute, Bangalore

B.Math (Hons.) III Year, First Semester

Semestral Examination, Back paper

Complex Analysis

Time: 3 hours | Instructor: B.Bagchi

Maximum marks: 100

1. (a)If $\gamma : [0,1] \to \mathbb{C}^*$ is a closed path then show that there is a path $\delta : [0,1] \to \mathbb{C}$ such that $\gamma = \exp \circ \delta$

   (b) For any closed path γ in $\mathbb{C}$ and $z \varepsilon \mathbb{C} \backslash tr(\gamma)$, define the index of z with respect to γ, and find an integral formula (with proof) for the index.

   [15]

2. Let Ω be a planar domain and $f : \Omega \to \mathbb{C}$ continuous. If $\int\limits_{\gamma} f = 0$ for all closed paths γ in Ω then show that f is holomorphic on Ω. Give an example to show that the converse statement is false, in general. [10]

3. For $z \varepsilon \mathbb{C}$, let $G(z) = z e^{\gamma z} \prod\limits_{n=1}^{\infty} (1 + \frac{z}{n}) e^{-\frac{z}{n}}$, where γ is a constant.

   (a) Show that the above product converges locally uniformly on $\mathbb{C}$. Deduce that G is an entire function.

   (b) Find all the zeros of G. (Prove that your list of zeros is complete.)

   (c) Show that there is a choice of the constant γ for which $G(1) = 1$ and $G(z+1) \equiv \frac{1}{z} G(z)$. [20]

4. Let Ω be a simply connected planar domain with $0 \varepsilon \Omega$. Let $r_o > 0$ be a constant. Let $\mathcal{F} = \{f \varepsilon H(\Omega) : f(0) = 0 \text{ and } f(z) \neq \pm 1 \text{ for all } z \varepsilon \Omega\}$, $\mathbb{G} = \{g \varepsilon H(\Omega) : g(0) = \frac{1}{3} \text{ and } g(\Omega) \text{ does not contain any disc of radius } r_0\}$.

   Let $T : H(\Omega) \to H(\Omega)$ be defined by $(Tf)(z) = \cos(\pi \cos(\pi f(z))), z \varepsilon \Omega, f \varepsilon H(\Omega)$.

   (a) Prove that T is continuous (with respect to the usual topology on $H(\Omega)$).

   (b) Use Block's Theorem to show that $\mathbb{G}$ is a precompact subset of $H(\Omega)$.

   (c) Show that there is a choice of r_0 for which $T(\mathbb{G}) \supseteq \mathcal{F}$. Deduce that $\mathcal{F}$ is a precompact subset of $H(\Omega)$.

   [5+15+20=40]

5. If f is a non-constant holomorphic function on a planar domain Ω, then show that there is a function $k : \Omega \to \{1, 2, 3, \cdots\}$ such that for each $z \varepsilon \Omega$, f is exactly $k(z)-$ to $-$ one on some punctured neighbourhood of z. [15]