Final - Complex Analysis (2021-22)
Time: 3 hours.

Attempt all questions, giving proper explanations.
You may quote any result proved in class without proof.
`The exam is open-book`

1. Show that if both f and $\bar{f}$ are holomorphic in a region D then f is a constant. **[3 marks]**

2. For $a, b \in \mathbb{C}$, maps of the form $z \mapsto z+b$ are called translations, maps of the form $z \mapsto az$ are called dilations, and $z \mapsto \frac{1}{z}$ is an inversion. Show that every fractional linear transformation is a composition of dilations, translations and inversions. **[5 marks]**

3. Let $f(z)$ be a bounded analytic function on the right half-plane $\{\text{Re } z > 0\}$. Suppose that $f(z)$ extends continously to the imaginary axis and satisfies $|f(iy)| \leq M$ for all points iy on the imaginary axis. Show that $|f(z)| \leq M$ for all z in the right half-plane. **[5 marks]**
HINT: Consider $(z+1)^{-\epsilon} f(z)$ on a large semidisk, for an appropriate ϵ.

4. Compute
$$\int_{|z|=1} \frac{dz}{z^2(z^2-4)e^z},$$
where the integral is over a circle centered at the origin of radius 1 in the counterclockwise direction. **[3 marks]**

5. Consider real-valued continuously differentiable functions $P(x, y)$, $Q(x, y)$ defined on the unit disc $\mathbb{D}$. Suppose
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \text{for } (x, y) \in \mathbb{D}.$$
Show that there is a function $h : \mathbb{D} \to \mathbf{R}$ such that
$$\frac{\partial h}{\partial x} = P, \qquad \frac{\partial h}{\partial y} = Q \quad \text{on } \mathbb{D}. \quad \textbf{[5 marks]}$$
HINT: You might need Green's theorem which states that for any bounded region $D \subset \mathbf{R}^2$ whose boundary ∂D consists of a finite number of disjoint piecewise smooth closed curves, and for P, Q real-valued continuously differentiable functions on $D \cup \partial D$ we have
$$\int_{\partial D} Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

6. Use contour integration to show that
$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{(1+x^2)^2} dx = \frac{\pi}{2}(1+2\pi\xi)e^{-2\pi\xi}$$
for $\xi > 0$. **[6 marks]**

7. Suppose that $f(z)$ is analytic in a bounded region Ω, and suppose that $|f(z)| \leq M$ for all $z \in \Omega$.
   (a) Show that for each $\delta > 0$ and $m \geq 1$
   $$\left| f^{(m)}(z) \right| \leq \frac{m! M}{\delta^m},$$
   for all $z \in \Omega$ whose distance from $\partial\Omega$ is at least δ. **[5 marks]**
   (b) Suppose $\{f_k(z)\}$ is a sequence of analytic functions on Ω that converges uniformly to $f(z)$ on Ω. Show that for each m the derivatives $f_k^{(m)}(z)$ converge uniformly to $f^{(m)}(z)$ on each subset of Ω at a positive distance from $\partial\Omega$. **[5 marks]**

8. Let f be holomorphic on the punctured unit disc $\mathbb{D}\backslash\{0\}$, and let γ be the circle of radius $\frac{1}{2}$ around 0. Assume further that

$$|f(z)| \leq \frac{1}{|z|^{\frac{1}{2}}} \qquad \text{for } |z| \leq \frac{1}{3}.$$

Compute $\int_\gamma f(z)dz$. **[6 marks]**

9. (a) Show that the map $f(z) = \exp\left(\frac{\pi z}{2}\right)$ takes the horizontal strip $\{-1 < \text{Im } z < 1\}$ to the right half plane $\{\text{Re } z > 0\}$. **[3 marks]**

   (b) Find a conformal map of the vertical strip $\{-1 < \text{Re } z < 1\}$ onto the open unit disc $\mathbb{D}$. **[4 marks]**