Indian Statistical Institute

Final Examination 2019-2020

B.Math Third Year

Complex Analysis

Time : 3 Hours Date : 11.11.2019 Maximum Marks : 100 Instructor : Jaydeb Sarkar

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Note: (i) Answer all questions. (ii) $B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$. (iii) $C_r(z_0) := \{z \in \mathbb{C} : |z - z_0| = r\}$. (iv) $U \subseteq \mathbb{C}$ open. (v) $\text{Hol}(U) = \{f : U \to \mathbb{C} \text{ holomorphic }\}$. (vi) $\mathbb{D} = B_1(0)$ and $\mathbb{T} = C_1(0)$.

Q1. (10 marks) Determine the fractional linear transformation mapping 0 to 2, $2i$ to 0, and i to $\frac{3}{2}$.

Q2. (10 marks) Use Rouche's theorem to prove that any degree n polynomial has n zeros.

Q3. (10+5 = 15 marks) Let

$$f(z) = \frac{1}{\sin z}.$$

(a) Find all the poles and their residues of f. (b) Use the answer of part (a) to compute

$$\int_{C_{2019}(0)} \frac{dz}{\sin z}.$$

Q4. (15 marks) Show that there is no function $f \in \text{Hol}(\mathbb{D})$ that extends continuously to $\mathbb{T}$ such that $f(z) = \bar{z}$ for all $z \in \mathbb{T}$.

Q5. (15 marks) Let f be an entire function. Suppose that for each $w \in \mathbb{C}$, at least one coefficient in the power series expansion at w is zero. Prove that f is a polynomial.

Q6. (15 marks) Prove that meromorphic functions on the extended complex plane are rational functions.

Q7. (15 marks) Let $f \in \text{Hol}(\mathbb{D} \setminus \{0\})$. Show that if $|f(z)| \leq \log \frac{1}{|z|}$ for all $z \in \mathbb{D} \setminus \{0\}$, then $f \equiv 0$.

Q8. (15 marks) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic and assume that $f(\alpha) = 0 = f(\beta)$ for two different numbers α and β in $\mathbb{D}$. Prove that

$$|f(z)| \leq \left|\frac{\alpha - z}{1 - \bar{\alpha} z}\right| \left|\frac{\beta - z}{1 - \bar{\beta} z}\right| \qquad (z \in \mathbb{D}).$$