Indian Statistical Institute

Semestral Examination 2018-2019

B.Math Third Year   Complex Analysis   November 14, 2018   Instructor : Jaydeb Sarkar

Time : 3 Hours   Maximum Marks : 100

Notation: (i) $B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$. (ii) $Hol(B_r(z_0)) = \{f : B_r(z_0) \to \mathbb{C}$ holomorphic $\}$. (iii) $C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$. (iv) $\mathcal{Z}(f) =$ zero set of f. (v) $\epsilon > 0$. (vi) $\mathbb{H} =$ Upper half plane.

(1) (10 marks) Let $f : \mathbb{C} \to \mathbb{H}$ be a holomorphic function. Prove that f is constant.

(2) (10 marks) Let γ be a smooth closed curve in $\mathbb{C}$. Prove that the winding number of γ is identically zero on the unbounded component of $\mathbb{C} \setminus \gamma$.

(3) (10 marks) Prove that there is no function f that is holomorphic on $\{z \in \mathbb{C} : 0 < |z| < 1\}$, and f' has a simple pole at 0.

(4) (10 marks) Let $f \in Hol(B_{1+\epsilon}(0))$, $f(0) = 0$, $f'(0) = 1$, and suppose that $|f(z)| \leq 1$ for all $z \in C_1(0)$. Prove that $f(z) = z$ for all $z \in B_{1+\epsilon}(0)$.

(5) (15 marks) Let $f \in Hol(B_{1+\epsilon}(0))$, and suppose that $|f(z)| < 1$ for all $z \in C_1(0)$. Prove that the equation $f(z) = z^3$ has exactly three solutions (counting multiplicities) inside $B_1(0)$.

(6) (15 marks) Examine the nature of the singularities of the following functions and determine the residues at the singularities:

$$(a)\ \frac{1}{\sin z}, \qquad (b)\ \frac{1+z}{z}.$$

Use part (a) to find $\int_{C_1(0)} \frac{1}{\sin z}\, dz$.

(7) (15 marks) Prove that, except for the identity function, a holomorphic self-map of $B_1(0)$ has at most one fixed point.

(8) (15 marks) Let $f \in B_1(0)$ and $g \in Hol(\{z \in \mathbb{C} : |z| > r\})$ for some $r < 1$. Suppose that g admits a limit at infinity. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^n}$ be the corresponding Taylor and Laurent expansions centered at the origin. Prove that the series

$$\sum_{n=0}^{\infty} a_n b_n,$$

converges absolutely.

(9) (15 marks) Let $f \in Hol(B_1(0) \setminus \{0\})$. If f is one-to-one, then prove that 0 is either a pole or a removable singularity of f.