Indian Statistical Institute

Semestral Examination 2017-2018

B.Math Third Year Complex Analysis November 15, 2017 Instructor : Jaydeb Sarkar

Time : 3 Hours Maximum Marks : 100

(i) $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. (ii) $Hol(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \text{ holomorphic }\}$. (iii) $C(\overline{\mathbb{D}}) = \{f : \overline{\mathbb{D}} \to \mathbb{C} \text{ continuous }\}$. (iv) $\mathcal{Z}(f) =$ zero set of f.

(1) (10 marks) Let $f \in Hol(\mathbb{D})$, and assume that $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. If $f(0) = 0$, then prove that the series
$$\sum_{n=0}^{\infty} f(z^n),$$
converges absolutely and uniformly on $\{z \in \mathbb{C} : |z| \leq r\}$, $r < 1$.

(2) (10 marks) Let γ be a smooth closed curve in $\mathbb{C}$. Prove that the winding number of γ is identically zero on the unbounded component of $\mathbb{C} \setminus \gamma$.

(3) (10 marks) Prove that there is no branch of the logarithm on $\mathbb{C} \setminus \{0\}$.

(4) (10 marks) If $\alpha^4 + \alpha^3 + 1 = 0$, $\alpha \in \mathbb{C}$, then prove that $|\alpha| < \frac{3}{2}$.

(5) (15 marks) Let f be a meromorphic function on $\mathbb{C}$, and let
$$|f(z)| \leq \Big(\frac{|z|}{|z-1|}\Big)^{\frac{3}{2}}.$$
Prove that $f \equiv 0$.

(6) (15 marks) Let $\{f_n\}$ be a sequence in $C(\overline{\mathbb{D}}) \cap Hol(\mathbb{D})$. Suppose that f_n converges uniformly on $\partial\mathbb{D}$ to a function f. Prove that f can be extended to a function in $C(\overline{\mathbb{D}}) \cap Hol(\mathbb{D})$.

(7) (15 marks) Examine the nature of the singularities of the following functions and determine the residues at the singularities:
$$(a)\ \frac{1}{\sin\frac{1}{z}}, \qquad (b)\ \frac{e^{-z}}{z^2}.$$
Use part (b) to find
$$\int_{|z|=3} \frac{e^{-z}}{z^2}\, dz.$$

(8) (15 marks) Let $\{f_n\}$ be a sequence of entire functions, and assume that f_n converges uniformly to a non-identically zero function f on $\mathbb{C}$. If $\mathcal{Z}(f_n) \subseteq \mathbb{R}$ for all n, then prove that $\mathcal{Z}(f) \subseteq \mathbb{R}$.

(9) (15 marks) Let $f \in Hol(\mathbb{D})$, and assume that $|f(z)| < 1$ for all $z \in \mathbb{D}$. Prove that
$$\Big|\frac{f(z) - f(w)}{1 - f(z)\overline{f(w)}}\Big| \leq \Big|\frac{z - w}{1 - z\overline{w}}\Big|,$$
for all z and w in $\mathbb{D}$.