Indian Statistical Institute

Final Examination 2015-2016

B.Math Third Year

Complex Analysis

Time : 3 Hours   Date : 04.11.2015   Maximum Marks : 100   Instructor : Jaydeb Sarkar

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(i) Answer all questions. (ii) $B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$. (iii) $\mathbb{H}$ = upper half plane. (iv) $C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$. (v) $\mathbb{A}_{1,2}(0) = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

Q1. (15 marks) Let $f : \mathbb{C} \to \mathbb{H}$ be a holomorphic function. Prove that f is a constant.

Q2. (15 marks) Let $f : B_1(0) \to B_1(0)$ be a holomorphic function. Let $\alpha \in B_1(0)$ and $f(\alpha) = 0$. Prove that $|f(0)| \leq |\alpha|$.

Q3. (15 marks) Let $g(z) = f(z^3)$ where $f \in \text{Hol}(\mathbb{C})$ and f is not identically zero. Prove that

$$\text{Res } [\frac{1}{g}; 0] = 0.$$

Q4. (15 marks) Prove that $f(z) = 2 - z - e^{-z}$ has one root in the right half plane.

Q5. (15 marks) Let $f \in \text{Hol}(\mathbb{C})$ and $f(0) = 0$, and $f'(0) = 1$ and suppose that $|f(z)| \leq 1$ for all $z \in C_1(0)$. Show that $f(z) = z$ for all $z \in \mathbb{C}$.

Q6. (15 marks) Use the residue theorem to compute the following integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}.$$

Q7. (15 marks) Prove there does not exist a branch of $\log(z^2 - 1)$ on $\mathbb{C} \setminus [-1, 1]$.

Q8. (15 marks) Prove or disprove (with justification):

(i) There exist $f \in \text{Hol}(\mathbb{C} \setminus \{0\})$ such that $f(z)^2 = z$ for all $z \in \mathbb{C} \setminus \{0\}$.

(ii) There exist $f \in \text{Hol}(\mathbb{A}_{1,2}(0))$ such that $f(z)^2 = z$ for all $z \in \mathbb{A}_{1,2}(0)$.