Indian Statistical Institute
First Semester Examination 2005-2006
B.Math. (Hons.) III Year
Complex Analysis

Time: 3 hrs | Date: 22-11-2005 | Total Marks: 60

Answer as many questions as you can. The maximum you can score is 55. All questions carry equal marks. You may use your class-room notes in the exam. However you cannot use textbooks.

1. f_n holomorphic on $\Omega = \{z : |z| < 1\}, f_n(0) = 0$ and $f_n' \to g$ uniformly on compact subsets of Ω. Show that there exists f, holomorphic on Ω, such that $f_n \to f$ uniformly. How are f and g related ?

2. f is nonconstant and holomorphic on $\Omega = \{z : |Im\ z| < 1\}$ and continuous on $\bar{\Omega} = \{z : |Im\ z| \leq 1\}$. Also $|f(z)| < |f(Im\ z)|$ for $z \in \bar{\Omega}$. Prove that f is bounded on $\bar{\Omega}$. Identify the points of $\bar{\Omega}$ on which $|f|$ may have a maximum.

3. Let $f_N(z) = \sum\limits_{n=1}^{N} \frac{n^2 z^{n-1}}{n^2+z^2}$. Prove that f_n converges to a holomorphic function in $\Omega = \{z : |z| < 1\}$. If γ is a circle of radius $0 < r < 1$, centred at 0, what is $\int\limits_{\gamma} \frac{f(z)}{z}\, dz$?

4. f is an entire function such that $\frac{f(z)z^3}{1+z^2}$ is bounded on $\mathbb{C}\backslash\{-i, i\}$. What conclusions can you draw about f ?

5. Evaluate $\int\limits_{-\infty}^{\infty} \frac{t^2}{1+t^4} dt$ using contour integration.

6. (a) Let $w = \frac{z-i}{z+i}$ be the conformal map of the open upper half plane $\mathbb{H}$ onto the open unit disc $\mathbb{D}$. Show that the pull back of the Poincare metric λ of the unit disc $\mathbb{D}$ under w is $2\frac{|dz|}{|y|}$, where $z = x + iy \in \mathbb{H}$.

   (b) Let f be a holomorphic function on $\{0 < |z| < r\}$ omitting the two values 0 and 1 from its range. Let φ be a metric on $\mathbb{C}\backslash\{0, 1\}$ whose Gaussian curvature is at most -4. Show that the length of $f(C_\epsilon)$, computed with respect to the metric φ, goes to zero as $\epsilon \to 0$, where C_ϵ is the circle of radius ϵ centered at 0.

   (Hint: Assume $r = 1$ and consider $g(w) = f(e^{iw})$, where w is as in part (a).)