Classical Mechanics
ISI B.Math
Midterm Exam : September 21, 2023

Total Marks: 70
Time : 3 hours

Answer all questions:

1.($\mathbf{Marks = 2 \times 7 = 14}$)

In this question you need to write down the correct option. No explanation is necessary. There is only one correct option.

(i)A particle is moving under the influence of a force $\mathbf{F} = (\frac{a \sin t}{r^2} - \frac{b \cos t}{r^3})\hat{\mathbf{r}}$, where a and b are constants. Which of the following statements are not true about the motion of the particle ?

(a) Angular momentum about the origin is conserved
(b) Total mechanical energy is conserved
(c) $\nabla \times \mathbf{F} = 0$.
(d) The work done by the particle in moving from one point to another is independent of path.
(e) The motion remains confined to a plane.

(ii) An undamped harmonic oscillator of mass m and angular frequency ω_0 moves in one dimension along the x- axis. If we plot x vs the linear momentum p_x (a phase space plot) for a given set of initial conditions, the resulting curve will be

(a) closed
(b) open
(c) can be closed or open depending upon the initial conditions.

(iii) S is an inertial frame S' is another inertial frame moving with velocity $\mathbf{v}$ with respect to S. For which of the following quantities will an observer in S and one is S' disagree on their measurements ?

(a) The gravitational force $\mathbf{F}$ acting on a particle of mass m_1 due to another particle m_2 .
(b) The mutual potential energy $U(\mathbf{r})$ of the above two particles interacting though a gravitational force where $\mathbf{r}$ is the relative position vector between the two particles.
(c) The total mechanical energy (kinetic + potential) of the two particles mentioned in the two above options.

(iv) A particle is moving in three dimensions under the influence of a potential $U(r) = \frac{1}{2}kr^2$, where k is a positive constant. Which of the statements about the motion of the particle is false ?

(a) A possible trajectory for the particle is a circular orbit.
(b) The particle can have bounded or unbounded motion depending on its total energy.
(c)The radius vector of the particle sweeps out equal areal in equal times
(d) The total energy of the particle is conserved.

(v) A ball bearing of mass m is released from rest in a vertical column of castor oil which exerts a retarding force equal to $-mkv$ on the ball bearing. which of the following expressions can correctly describe its velocity at time t ?

(a) $v = \frac{g}{k}(1 - e^{-kt})$
(b) $v = \frac{g}{k}(1 - e^{kt})$
(c) $v = \frac{g}{k}e^{-kt}$
(d) $v = \frac{mg}{k}(1 - e^{-kt})$

(vi) A cricket ball of mass m and a bowling ball of mass M (M$\gg m$) are simultaneously projected from the ground with a speed v at an angle α with the horizontal. Ignore air resistance. Which of the following statements is true ?

(a) The maximum height reached by the cricket ball is greater than that of the bowling ball.
(b) The maximum height reached by the cricket ball is equal that of the bowling ball
(c) The maximum height reached by the cricket ball is greater than that of the bowling ball
(d) It is not possible to conclude (a) , (b) or (c) on the basis of the data given
.

(vii) A particle of mass m and charge q is moving under the influence of a constant magnetic field in the z direction. Which of the following statements is true ?

(a) Total linear momentum is conserved.
(b) Kinetic energy is conserved.
(c) The charged particle always moves in a circular trajectory

2. (**Marks = 7 + 7 = 14**)

a) A particle with polar coordinates r, θ which are functions of time t is moving in a plane. The velocity and acceleration of the particle can be written in plane polar coordinates as $\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\theta}$ and $\mathbf{a} = a_r\hat{\mathbf{r}} + a_\theta\hat{\theta}$. Find $v_r, v_\theta, a_r, a_\theta$ as functions of $(r, \theta, \dot{r}, \dot{\theta}, \ddot{r}, \ddot{\theta})$

b) A racing car moves on a circular track of radius b. The car starts from rest and its *speed* increases at a constant rate α. Find the angle between the velocity and acceleration vectors at time t.

3. (**Marks = 3 + 3 + 4 + 4 = 14**)
A particle P of unit mass moves on the positive x- axis under the force field

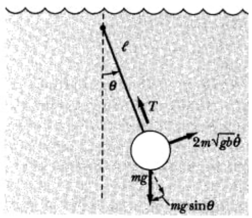
$$F = \frac{36}{x^3} - \frac{9}{x^2}$$

where $x > 0$.
(a) Find the potential $U(x)$ corresponding to this force and make a rough plot of $U(x)$ vs x.
(b) Show that the motion of P consists of either(i) periodic oscillation between two extreme points or (ii) an unbounded motion with one extreme point, depending upon the value of total energy.
(c) Initially, P is projected from the point $x = 4$ with speed 0.5. Show that P oscillates between two extreme points and find the period of the motion. You may make use of the formula

$$\int_a^b \frac{x\,dx}{[(x-a)(b-x)]^{\frac{1}{2}}} = \frac{\pi(a+b)}{2}$$

(d) Show that there is a single equilibrium position for P and that it is stable. Find the period of small oscillations about this point.

4.(**Marks = 8 + 6 = 14**)



Consider a pendulum of length l and a bob of mass m at its end. The pendulum makes an angle θ with the vertical. The pendulum is moving through oil with θ decreasing. The massive bob undergoes small oscillations, but the oil retards the bob's motion with a resistive force proportional to the speed with $F_{res} = 2m\sqrt{\frac{g}{l}}(l\dot{\theta})$. The bob is initially pulled back at $t = 0$ with $\theta = \alpha$ and $\dot{\theta} = 0$.

(a) Find the angular displacement θ and velocity $\dot{\theta}$ as a function of time. Is the total mechanical energy conserved ? Explain.
(b) Now the pendulum and bob are removed from the oil and the pendulum is allowed to oscillate with the same initial conditions. Find $\theta(t)$. Make a rough plot of θ versus $\dot{\theta}$ under these conditions.

5. (**Marks = 3 + 3 + 4 + 4 = 14**)

(a) A particle of mass m moves under the influence of a central force $\mathbf{F}(\mathbf{r})$. Show that angular momentum is conserved and that the orbit lies in a plane.

(b) Write down the equations of motion for the particle in polar coordinates . (Hint: your task will be simplified if you use the results of problem 1 a))

(c) Using the result of part (b), and setting $u = \frac{1}{r}$, show that

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2}F(1/u)$$

where L is the angular momentum of the particle.

(d) If the particle is moving in a trajectory such that $r\theta$ = constant, find the potential energy of the particle.