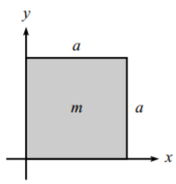
Physics I
ISI B.Math
Final : November 22, 2023

Total Marks: 50
Time : 3 hours

Answer all questions:

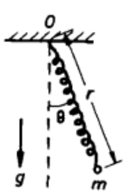


1. (**Marks = 3 + 2 + 5**)

(a) Find the inertia tensor of a uniform square of mass m, mass per unit area ρ and side a when one corner is at the origin and the two adjacent sides lie along the positive $x_1(x)$ and $x_2(y)$, coordinate axes. Recall that the form of the inertia tensor is given by $I_{ij} = \rho \int_A \left(\delta_{ij} \sum_k {x_k}^2 - x_i x_j\right) dA$ where $dA = dx_1 dx_2$ and where A is the area of the body and $i, j = 1, 2, 3$.

(b) If the above square rotates about the x_1 axis with an angular velocity $\omega = \omega \mathbf{e_1}$, where $\mathbf{e_1}$ is a unit vector in the x_1 direction, find the component L_1 of the angular momentum about the origin along the same direction. Will the angular momentum vector point in the same direction ?

(c) Find the three principal moments of inertia and the three orthogonal unit vectors along the principal axes corresponding to the principal moments of inertia.



2. (**Marks = 3 + 4 + 3**)

A massless spring of unstretched length l_0 has a point mass m connected to one end and the other end fixed so that the spring hangs in the gravity field as shown in the figure.The motion of the system is only in one vertical plane.

(a) Write down the Lagrangian in terms of the generalized coordinates θ and r.

(b) Find Lagrange's equations using variables θ, $\lambda = \frac{(r-r_0)}{r_0}$ where r_0 is the rest length (hanging with mass m). Use $\omega_s^2 = \frac{k}{m}$ and ${\omega_p}^2 = \frac{g}{r_0}$.

(c) Solve the equations of motion under the approximation of small $\lambda, \theta, \dot{\theta}, \dot{\lambda}$, with the initial conditions $\theta = 0, \dot{\lambda} = 0, \lambda = A, \dot{\theta} = \omega_p B$ at $t = 0$, where A and B are constants.

3. (**Marks = 1 + 5 + 4**)

A particle of mass m is moving under the influence of a central force $\mathbf{F}(\mathbf{r}) = -\frac{dU(r)}{dr}\hat{\mathbf{r}}$ where $U(r)$ is the potential corresponding to the force.

(a) Write down the Lagrangian for the system in two-dimensional polar coordinates (r, θ).

(b) Identify the cyclic coordinate and write down the generalized momentum corresponding to it. Write down the Lagrange equations of motion for this system and show that the generalized momentum you identified previously is conserved. Which physical conservation law does this correspond to ? Which symmetry is associated with this conservation law ?

(c) Given that the force $F(r)$ is such that it makes the particle move in a logarithmic spiral orbit $r = ke^{\alpha\theta}$ where α and k are constants, use this information and the equations of motion to find $\theta(t)$ and $r(t)$ in terms of the angular momentum l and the given constants k and α. Your answer will also involve an arbitrary constant which can be fixed from initial conditions.

4. (**Marks** $= \mathbf{4} + \mathbf{4} + \mathbf{2}$)

(a) Show that if all three principal moments of inertia are equal $(I_1 = I_2 = I_3 = I)$, then ANY axis (through the chosen origin) in space is also a principal axis and its moment of inertia is also I.

(b) What must be the ratio of the height to radius of a cylinder so that every axis is a principal axis (with the CM as the origin)?

(c) A plumb bob or a plumb line is an instrument which consists of a weight with a pointed tip at the bottom connected to a string. It is often used in surveying and constructing buildings to determine the direction of the true vertical. When the plumb bob is at rest and suspended by the string, it is supposed to point toward the centre of the earth, in the direction of the local gravitational acceleration. However, if we take into account the rotational motion of the earth, the direction of the bob will deviate from the true vertical. Explain why this happens and state in which direction the plumb bob will actually point.

5. (**Marks** $= \mathbf{5} \times \mathbf{2}$)

State whether the following statements are true or false with a very brief (one or two lines) explanation.

i) A particle of mass m is moving in one dimension under the influence of a force $F = -kx.L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + a\dot{x} - bt$ where a and b are constants is a valid Lagrangian for the system.

ii) A point mass m travels in a circle of radius R with centre at the point $(0, 0, z_0)$ with the plane of the circle parallel to the $x - y$ plane with an angular velocity $\omega\hat{\mathbf{z}}$.The angular momentum vector about the origin points in the same direction as the angular velocity.

iii) A rigid body always has exactly 6 degrees of freedom.

iv) A particle of mass m moving under the influence of a force $\mathbf{F} = \frac{k}{r^2}\hat{\mathbf{r}}$, where $k > 0$ can move in an elliptical orbit.

v) In the Northern Hemisphere, a particle projected in a horizontal plane will be deviated to the right of the particle's motion due to the effect of the Coriolis force. In the Southern Hemisphere the deflection will be towards the left.