February 24th, 2023
TAS - Midterm - Semester II

Instructions: Closed Book Exam. Please write clear and precise answers.

1. Let $\alpha > 0$. Consider the graph $\mathbb{Z}_+$ with weights $\mu^{\alpha}_{n,n+1} = \alpha^n$.
   (a) (5 points) Show the graph $(\mathbb{Z}_+, \mu^{\alpha})$ has controlled weights. Does it have bounded weights ?
   (b) (10 points)Show that the graph is recurrent if and only if $\alpha \leq 1$.

2. Consider Γ to be the join of two copies of $\mathbb{Z}^3$ at their origins. Write $\mathbb{Z}^3_{(i)}, i = 1, 2$ the two copies, and 0_i for their origins. Let
$$F = \{X \text{ is ultimately in } Z^3_{(1)}\}$$
and let $h(x) = \mathbb{P}^x(F)$.
   (a) (3 points) Show that h is harmonic,
   (b) (4 points) Show that $h(x) \geq \mathbb{P}^x(X \text{ never hits } 0_1)$ for $x \in \mathbb{Z}^3_{(1)}$.
   (c) (4 points) Show that $h(x) \leq \mathbb{P}^x(X \text{ hits } 0_2)$ for $x \in \mathbb{Z}^3_{(2)}$.
   (d) (4 points) Decide whether Γ has the Liouville Propety: All bounded harmonic functions on Γ are constant.

3. Let X_n be a random walk on $\mathbb{Z}$.
   (a) (3 points) Show that $L = \sup\{n \geq 1 : X_n = 1\}$ is not a stopping time.
   (b) (3 points) Show that $F = \inf\{n \geq 1 : X_n \in \{0, 4\}\}$ is a stopping time. Can you find the distribution of X_F ?
   (c) (3 points) Show that the return time T_i to a state $i \in S$ is a stopping time.
   (d) (3 points) Let $T_a = \inf\{n \geq 1 : X_n = a\}$. Show that T_a is a stopping time and the 'inf' in T_a is actually a minimum almost everywhere.
   (e) (3 points) (Reflection Principle) Suppose $M_n = \max_{0 \leq i \leq n} X_i$. Show that for $X_0 = 0$ and $a > 0$,
$$P(M_n \geq a, X_n < a) = P(M_n \geq a, X_n > a).$$
   (Hint: Apply the strong markov property at T_a and symmetry of the distribution of the Bernoulli trials.)

4. An urn contains R red and G green balls. At each time we draw a ball from the urn, then replace it, and add C balls of the colour drawn. Let $\{X_n\}_{n \geq 1}$ be the fraction of greem balls after the n-th draw. Show that X_n is a martingale.