Indian Statistical Institute, Bangalore
B. Math.
Third Year, First Semester
Analysis on Graphs

Mid-term Examination — Date : 23 September 2023
Maximum marks: 100 — Time: 10.00AM-1.00PM
Instructor: B V Rajarama Bhat

All the graphs considered are simple graphs, without loops or multiple edges.

(1) Let $S = \{1, 2, \ldots, n\}$ where n is a natural number. Consider a graph G where the vertex set of G is the collection of all subsets of S:

$$V(G) = \{A : A \subseteq S\}.$$

Two subsets A, B of S, form an edge of G if the number of elements in the symmetric difference

$$A \triangle B := (A \bigcup B) \bigcap (A \bigcap B)^c$$

is 1.

(i) Show that G is regular, that is the degrees of all the vertices are equal. Verify that the sum of degrees is twice the number of edges.

(ii) Write down the adjacency matrix and Laplacian matrix of G.

(iii) Write down an eigenvalue and an eigenvector for the adjacency matrix of G. [21]

(2) Let G_1, G_2 be two graphs. Then the Kronecker product of G_1, G_2 is the graph G with vertex set $V(G) = V(G_1) \times V(G_2)$ and edge set $E(G)$, where $\{(u, x), (v, y)\}$ with $(u, x), (v, y)$ in $V(G)$ is an edge iff $\{u, v\} \in E(G_1)$ and $\{x, y\} \in E(G_2)$. (i) Identify the adjacency matrix of G in terms of adjacency matrices of G_1, G_2. (ii) Suppose H is the graph with $V(H) = \{1, 2, 3\}$ and $E(H) = \{\{1, 2\}, \{2, 3\}\}$. Compute eigenvalues of the adjacency matrix of the Kronecker product of H with itself. [21]

(3) Let H be a graph with $V(H) = \{1, 2, 3, 4\}$

$$E(H) = \{\{i, j\} : i \neq j, \{i, j\} \neq \{1, 2\}, \{i, j\} \neq \{1, 3\}, i, j \in V(H)\}.$$

(i) Describe/draw all the spanning trees of H. (ii) Verify the matrix-tree theorem for this example. [21]

(4) Let G_1, G_2 be the graphs given by $V(G_1) = \{x, y, z\}\}$. $E(G_1) = \{\{x, y\}, \{x, z\}, \{y, z\}\}$. $V(G_2) = \{a, b\}$, $E(G_2) = \{\{a, b\}\}$. Compute the energy of the Cartesian product $G_1 \times G_2$. [21]

(5) Let L be the Laplacian matrix of a graph G with n vertices. Show that the rank of L is $n - k$ where k is the number of connected components of G. [21]