Indian Statistical Institute, Bangalore
B. Math.
Third Year, First Semester
Analysis on Graphs

Final Examination — Date : November 20, 2023, 2023
Maximum marks: 100 — Time: 3 hours
Instructor: B V Rajarama Bhat

In the following G is a simple graph (undirected and without loops), with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. J denotes $n \times n$ matrix whose entries are all equal to 1.

(1) Suppose $i, j \in V(G)$. Show that there exist real numbers $\mu_1, \mu_2, \ldots, \mu_n$ with following property: $\sum_{i=1}^{n} \mu_i = 0$ and given any $j, k \in V(G)$ there exist scalars $s_1, s_2, \ldots s_n$ such that the number of walks of length m from j to k is given by,

$$\sum_{i=1}^{n} s_i \mu_i^m$$

for $m \geq 1$. (Hint: Recall that the number of walks are counted using powers of the adjacency matrix.) [21]

(2) Show that the number of spanning trees of a graph G with Laplacian L is equal to

$$\frac{1}{n^2}\det(L + J).$$

(You may carefully state the matrix-tree theorem and use it). [21]

(3) Show that the algebraic connectivity is monotone, that is, if $ac(G)$ denotes the second smallest eigenvalue of the Laplacian of G, and G is edge-disjoint union of two graphs H_1, H_2 on same vertex set, then

$$ac(G) \geq ac(H_1) + ac(H_2) \geq ac(H_1).$$

[21]

(4) Let K_m be the complete graph with m-vertices, where $m \geq 4$. Let L_m be the line-graph of K_m, that is, the edges of K_m are the vertices of L_m and there is an edge between them in L_m, if they have a common vertex in K_m. Show that L_m is strongly regular. Compute the parameters of strong regularity. Use this to compute the spectrum of L_m. Recall that the spectrum of a strongly regular graph with parameters (n, k, a, c) are k, λ_+, λ_- with multiplicities $1, m_+, m_-$ respectively, where

$$\lambda_{\pm} = \frac{1}{2}(a - c \pm \sqrt{\Delta}),$$

and

$$m_{\pm} = \frac{1}{2}[(n-1) \pm \frac{(n-1)(c-a) - 2k}{\sqrt{\Delta}}],$$

with $\Delta = (a - c)^2 + 4(k - c)$.

[21]

(5) Let $M = \{1, 2, 4, 5, 7, 8\}$ considered as a group under multiplication modulo 9. Let H be the Cayley graph of M with generating set $S = \{2, 5\}$. Write down adjacency, Laplacian and distance matrices of H. Compute the spectral radii of these matrices (Justify your claims). [21]