Indian Statistical Institute, Bangalore

B. Math. IIIrd Year Second Semester

Analysis IV

Mid-term examination | Date : 26-2-2019

Maximum marks: 100 | Time: 3 hours

(1) Consider $\mathbb{R}^2$ and $\mathbb{R}$ with usual Euclidean metric. Identify as to which of the following functions from $\mathbb{R}^2$ to $\mathbb{R}$ are continuous and which are uniformly continuous:

(i)$f_1((x,y)) = |x+y|$;

(ii) $f_2((x,y)) = x^2y$;

(iii) $f_3((x,y)) = \text{Max}\{x,y\}$.

Prove your claims. [15]

(2) Let $\{f_n\}_{n\geq 1}$ and $\{g_n\}_{n\geq 1}$ be sequences of real valued continuous functions on a metric space X converging uniformly to functions f,g respectively. Show that if f,g are bounded functions then, $\{f_ng_n\}_{n\geq 1}$ converges uniformly to fg. Show that this is not true in general without boundedness assumption on functions. [15]

(3) Consider the following differential equation on the real line:

$$y'(x) = x - 7y(x);\ y(0) = 0.$$

Show that there exists $h > 0$ such that this differential equation has unique solution in the interval $(-h,+h)$. Find such a solution. [15]

(4) Let $(X,d_1),(Y,d_2)$ be two metric spaces. Assume that (X,d_1) is compact. Suppose $f : X \to Y$ is a continuous function. Show that $f(X) = \{f(x) : x \in X\}$ is compact. If f is one to one, show that $f^{-1} : f(X) \to X$ is continuous. [15]

(5) Let $C([0,1],\mathbb{R})$ be the space of real valued continuous functions on the interval $[0,1]$ with supremum norm. Let S be the set of polynomials in $C([0,1],\mathbb{R})$, defined by

$$S = \{\sum_{k=0}^{10} a_kx^k : |a_k| \leq (k+1),\ 0 \leq k \leq 10\}.$$

Is S compact in $C([0,1],\mathbb{R})$? Prove your claim. [15]

(6) Let $X = C_0([0,1],\mathbb{R})$ be the space of continuous real valued functions on $[0,1]$ satisfying $f(0) = 0$. Show that the set of polynomials in X is dense in X with respect to supremum norm. [15]

(7) Let $Y = C([-1,1],\mathbb{R})$ be the space of real valued continuous functions on the interval $[-1,1]$ with supremum norm. Define $T : Y \to Y$ by

$$Tf(x) = (3x^2 - 2x + 11)f(x).$$

Show that T is a bounded linear map. Compute the operator norm of linear maps T and T^2. [15]