Indian Statistical Institute, Bangalore
B. Math. Third Year
Second Semester - Analysis IV

Mid-Semester Exam | Duration: 3 hours | Date : March 02, 2015

Answer any five, each question carries 8 marks, total marks: 40

1. Let $\mathcal{A}$ be a closed algebra of real continuous functions on a compact metric space X that separates points of X and nowhere vanishes on X.

   (a) If $f, g \in \mathcal{A}$, prove that $|f|$ and $\max\{f, g\}$ are in $\mathcal{A}$ *(Marks: 3+1)*.

   (b) For $f \in C_{\mathbb{R}}(X)$, $x \in X$ and $\epsilon > 0$, prove that there is a $g \in \mathcal{A}$ such that $g(x) = f(x)$ and $g(y) > f(y) - \epsilon$ for all $y \in X$.

2. Let $\Phi\colon C[0,1] \to C[0,1]$ be given by $\Phi(f)(x) = \int_0^x f(t)dt$.

   (a) Prove that Φ is continuous and $\Phi(B)$ is relatively compact for any bounded set $B \subset C[0,1]$ *(Marks: 5)*.

   (b) Is Φ a contraction? Does Φ have a unique fixed point? Justify your answers.

3. Let X be a complete metric space and $\phi\colon X \to X$ be a map.

   (a) If ϕ is a contraction, prove that ϕ has a unique fixed point $x \in X$ and $\lim_{n\to\infty} \phi^n(y) = x$ for all $y \in X$.

   (b) Suppose there is a sequence (a_n) such that $\sum a_n < \infty$ and $d(\phi^n(x), \phi^n(y)) \leq a_n d(x, y)$ for all $n \geq 1$ and all $x, y \in X$. Prove that ϕ has a unique fixed point $x \in X$ and $\lim_{n\to\infty} \phi^n(y) = x$ for all $y \in X$ *(Marks: 4)*.

4. (a) Discuss Implicit Function Theorem for F at $(2, -1, 2, 1)$ where $F\colon \mathbb{R}^{2+2} \to \mathbb{R}^2$ is given by $F(x, y, u, v) = (x^2 - y^2 - u^3 + v^2 + 4, 2xy + y^2 - 2u^2 + 3v^4 + 8)$.

   (b) Let X be a compact metric space and g be a continuous function on $\mathbb{C}$. Prove that $\phi\colon C(X) \to C(X)$ defined by $\phi(f) = g \odot f$ is continuous *(Marks: 4)*.

5. (a) Let f be a continuously differentiable map of an open set E of $\mathbb{R}^n$ into $\mathbb{R}^n$. If $f'(x)$ is invertible for every $x \in E$, prove that f is an open map *(Marks: 3)*.

   (b) Suppose f is a differentiable 2π-periodic function such that $f' \in \mathcal{R}[-\pi, \pi]$. Assume $f \sim \sum_{-\infty}^{\infty} c_n e^{inx}$. Prove that $\sum n^2 |c_n|^2$ and $\sum |c_n|$ converge.

6. Let $f \in \mathcal{R}[-\pi, \pi]$ be a 2π-periodic function and $f \sim \sum_{-\infty}^{\infty} c_n e^{inx}$.

   (a) If for some $x \in [-\pi, \pi]$, there is a $\delta > 0$ and $M < \infty$ such that for all $t \in (-\delta, \delta)$, $|f(x+t) - f(x)| \leq M|t|$, prove that $\lim_{N\to\infty} \sum_{-N}^{N} c_n e^{inx} = f(x)$.

   (b) Prove that $\lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t) - \sum_{-N}^{N} c_n e^{int}|^2 dt = 0$ *(Marks:4)*.

7. Let $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$. Prove that $f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ for all x. Deduce that $\sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and find $\sum \frac{1+2(-1)^n}{n^2}$.