Indian Statistical Institute

Mid-Semestral Examination 2011-2012

B.Math Second Year

Analysis IV

Time : 3 Hours Date : 29.02.2012 Maximum Marks : 100 Instructor : Jaydeb Sarkar

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Answer all questions.

Q1. (10 marks) Prove that the closed unit ball of $C[0,1]$ is not compact (under the usual uniform metric).

OR

Let $\{f_n\}$ be a sequence of functions in $C[0,1]$ with $f_n(0) = 0$ and $|f_n(x) - f_n(y)| \leq |x - y|$ for all $x, y \in [0,1]$ and $n \in \mathbb{N}$. Prove that $\{f_n\}$ has a uniformly convergent subsequence.

Q2. (15 marks) Let X be a compact metric space and $\mathcal{F} \subseteq C(X, \mathbb{R}^n)$ an equicontinuous family. Prove that if $\mathcal{F}$ is pointwise bounded (that is, for each $x \in X$, there exists $M_x > 0$ such that $\|f(x)\| \leq M_x$ for all $f \in \mathcal{F}$) then $\mathcal{F}$ is uniformly bounded (that is, $\|f(x)\| \leq M$ for all $f \in \mathcal{F}$ and $x \in X$ for some $M > 0$).

Q3. (15 marks) Let X be a compact metric space and $\mathcal{A}$ a closed subalgebra of $C(X, \mathbb{R})$ with $1 \in \mathcal{A}$. Let $f \in \mathcal{A}$ be a positive function (that is, $f \geq 0$ on X). Prove that $\sqrt{f} \in \mathcal{A}$. Also prove that $\{|f| : f \in \mathcal{A}\} \subseteq \mathcal{A}$.

Q4. (20 marks) Let X be a compact metric space and $T : X \longrightarrow X$ a map such that $d(Tx, Ty) < d(x, y)$ for all $x \neq y$ in X. Prove that T has a unique fixed point.

[Hint : If so, then $d(Tx, x) = 0$ for some $x \in X$.]

Q5. (10 marks) Let N be a fixed natural number. Consider the trigonometric polynomial $p(x) = c_0 + c_1 e^{ix} + c_2 e^{2ix} + \ldots + c_N e^{iNx}$ where $c_n \in \mathbb{R}$ and $\sum_{n=0}^{N} c_n^2 = 1$. Prove that

$$\int_{-\pi}^{\pi} |p(x)|\, dx \leq 2\pi.$$

Q6. (10 + 5 + 5 = 20 marks) Let $f \in \mathcal{R}[-\pi, \pi]$ with $f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$ and $s_N(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$, the N-th partial sum of the Fourier series of f.

(a) Given any trigonometric polynomial $p_N(x) = \sum_{n=-N}^{N} c_n e^{inx}$, prove that

$$\|f - s_N\| \leq \|f - p_N\|.$$

[$\|g\| := (\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(x)|^2\, dx)^{\frac{1}{2}}$, for all $g \in \mathcal{R}[-\pi, \pi]$.]

(b) Prove that $\sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 \leq \|f\|^2$.

(c) Prove or disprove the following statement - "There exists an integrable function $f \in \mathcal{R}[-\pi, \pi]$ whose Fourier series is the formal sum $\sum_{n=-\infty}^{\infty} e^{inx}$."

Q7. (10 marks) Let X be a compact metric space. Use the Stone-Weierstrass theorem to prove that $C(X, \mathbb{R})$ is separable.