B-III Analysis IV Back paper examination 2017.

Answer all the 7 questions. Question 1-6 are worth 15 points and the 7th-10 points.

If you are using any result proved in the class, you need to state it correctly. If the answer is an immediate consequence of the result quoted by you, you need to prove the result quoted by you.

1. Let (X, d) and (Y, ρ) be metric spaces. On the product space $X \times Y$, consider the function $r((x_1, y_1), (x_2, y_2)) = \sqrt{d(x_1, x_2)^2 + \rho(y_1, y_2)^2}$. Show that r is a metric.

2. Let (X, d) be a metric space and let $A \subset X$ be a compact set with more than 2 points. Show that there exists $a, b \in A$ such that the diameter of A, $dia(A) = d(a, b)$.

3. Let $\mathcal{F} = \{f \in C([0, 1]) : sup_{t\in[0,1]}|f(t)| \leq 1\}$. Show that $\mathcal{F}$ is not an equicontinuous set.

4. Let $T = \{z : |z| = 1\}$ and let $M = \{f \in C(T) : f(i) = 0\}$. Let $\epsilon > 0$, $f \in M$. Show that there is a polynomial p with $p(i) = 0$ and $sup_{z\in T}|f(z) - p(z)| \leq \epsilon$.

5. State and prove the Riemann-Lebesgue Lemma.

6. Let $f : [a, b] \to R$ be a Riemann integrable function. Let $\{\psi_n\}_{n\geq 1}$ be an orthonormal sequence on $[a, b]$. Suppose the Cesaro sums corresponding to the Fourier series of f converge uniformly to f. Show that $\int_a^b |f(t) - \psi_n(t)|^2 \, dt \to 0$.

7. Let $f : R \to R^3$ be such that f is differentiable and $f(R) \subset \{x \in R^3 : \|x\| = 1\}$. With respect to the dot product in R^3, Show that for every $t \in R$, $f(t).f'(t) = 0$.