Indian Statistical Institute, Bangalore
B. Math (Hons.) Third Year
Second Semester - Analysis IV

Semestral Exam — Date: 11th May 2022
Maximum marks: 50 — Duration: 3 hours

Answer any five, each question carries 10 marks

1. (i) Prove that $C(X)$ is separable for a compact metric space X (**Marks: 5**).

   (ii) Prove that $C[0,1]$ has no open set whose closure is compact.

2. (i) Let X be a compact metric space. For $r > 0$, let $E_r = \{f \in C(X) \mid |f(x) - f(y)| \leq rd(x,y) \quad for \quad all \quad x, y \in X\}$. Let $A \subseteq E_r$. Prove that $\overline{A}$ is compact if and only if $\{f(z) \mid f \in A\}$ is bounded for some $z \in X$.

   (ii) If X is a compact metric space and $\mathcal{A}$ is a closed subalgebra of $C_{\mathbb{R}}(X)$ that separates points of X, prove that either $\mathcal{A}$ nowhere vanishes or there is a $x_0 \in X$ such that $\mathcal{A} = \{f \in C_{\mathbb{R}}(X) \mid f(x_0) = 0\}$ (**Marks: 5**).

3. (i) Let E be an open subset of $\mathbb{R}^n$ and $f \colon E \to \mathbb{R}^n$ be a C^1-funcion. Prove that $f(\{x \in E \mid f'(x) \quad \text{is} \quad \text{invertible} \quad \})$ is open in $\mathbb{R}^n$.

   (ii) State and prove contraction mapping principle (**Marks: 5**).

4. (i)Find $\sum_{k=1}^{\infty} \frac{sin(2k-1)x}{2k-1}$ for any $x \in (-\pi, 0)$ using Fourier Series (**Marks: 5**).

   (ii) Describe a method of finding total variation of a differentiable function $f \colon [0,1] \to \mathbb{R}$ such that f' is 0 at only one point.

5. (i) Prove that Fourier series of any 2π-periodic bounded function that is monotonic in $[-\pi, \pi)$ converges (**Marks: 5**).

   (ii) Determine the Fourier coefficient of f defined by $f(x) = |x|$ for $|x| \leq 2$ and $f(x+4) = f(x)$ for all $x \in \mathbb{R}$.

6. (i) Let f and g be of bounded variation on $[a,b]$. Prove that $rf + sg$ and fg are also functions of bounded variation on $[a,b]$ for any constants r and s.

   (ii) State and prove Riemann-Lebesgue Lemma (**Marks: 5**).

7. (i) Let $f \sim \sum c_n e^{inx}$. Suppose $\sum n^2 |c_n|^2 < \infty$. Prove that $\sum c_n e^{inx}$ converges. Further if f is continuous at some point x, prove that $f(x) = \sum c_n e^{inx}$.

   (ii) Let f be a 2π-periodic continuously differentiable function and $\int_{-\pi}^{\pi} f = 0$. Prove that $\int_{-\pi}^{\pi} |f'|^2 \geq \int_{-\pi}^{\pi} |f|^2$ and the equality occurs if and only if $f(x) = acosx + bsinx$ for some constants a and b (**Marks: 5**).