Indian Statistical Institute
Final Examination 2020-2021
Analysis IV, B.Math Third Year
Date : 12.05.2021 Maximum Marks : 100 Duration : 3 Hours
Instructor : Jaydeb Sarkar (jaydeb AT gmail DOT com)

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(1) (10 marks) True or false (with justification): If $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuously differentiable bijective function, then F^{-1} is differentiable.

(2) (10 marks) True or false (with justification): $\{\frac{x^2}{x^2+(1-nx)^2}\}_{n\geq 1}$ is equicontinuous on $[0, 1]$.

(3) (10 marks) Let $F(x, y) = (x^2 - y^2, 2xy)$, $(x, y) \in \mathbb{R}^2$. Determine the points of $\mathbb{R}^2$ at which F is locally invertible with continuously differentiable inverse function. Also determine whether F has a continuously differentiable inverse defined on all of $\mathbb{R}^2$.

(4) (15 marks) Let $\{f_n\}$ be a sequence of differentiable functions on $[0, 1]$, and let $0 < t_0 < 1$. Suppose $\{f'_n\}$ is uniformly bounded and $\{f_n(t_0)\}$ is bounded. Prove that there exists a uniformly convergent subsequence of $\{f_n\}$.

(5) (15 marks) Let $C[0, 1]$ denote the space of all real-valued continuous functions defined on $[0, 1]$, and let $f, g \in C[0, 1]$. Suppose $f(x) < g(x)$ for all $x \in [0, 1]$. Prove that there exists a polynomial p such that

$$f(x) < p(x) < g(x) \qquad (\forall x \in [0, 1]).$$

(6) (15 marks) Compute

$$\lim_{n\to\infty} \int_0^1 (\cos x)(\cos(nx + n^3))^2 \, dx.$$

(7) (5+10+5+5=25 marks) Let f be a 2π-periodic function, and suppose $f(x) = x^2$, $x \in (-\pi, \pi)$.

(a) Sketch the graph of $y = f(x)$ from $x = -4\pi$ to $x = 4\pi$.

(b) Compute the Fourier series of f.

(c) Use part (b) to compute the value of $\sum_{n\geq 1} \frac{1}{n^2}$.

(d) Use part (b) to find the sum of the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$, $x \in (0, \pi)$.