Indian Statistical Institute, Bangalore
B. Math. IIIrd Year Second Semester
Analysis IV

Final examination — Date : 02-05-2019
Maximum marks: 100 — Time: 3 hours

(1) Let (X, d) be a compact metric space. Show that (X, d) is complete and totally bounded. Show that a complete and bounded metric space need not be compact. [15]

(2) Let $Y = C_{\mathbb{R}}[0,1]$ be the space of real valued continuous functions on [0,1]. Determine as to whether the following sets are equicontinuous or not and prove your claims.
  (a) $S_1 = \{f \in Y : f(\frac{1}{2}) = 0\}$;
  (b) $S_2 = \{f \in Y : 2 \leq f(t) \leq 3, \ \forall t \in [0,1]\}$;
  (c) $S_3 = \{f \in Y : \int_0^1 f(t)dt = 1\}$.

[15]

(3) Let (X, d) be a metric space. Let $T : X \to X$ be a strict contraction. Fix $x \in X$. Show that there exists $r > 0$ such that

$$T(y) \in B_r(x), \ \forall y \in B_r(x);$$

where $B_r(x) = \{z \in X : d(z, x) < r\}$. [15]

(4) Let A, B be closed disjoint subsets of $[0, 1]$. Given ϵ such that $0 < \epsilon < 1$, show that there exists a polynomial p such that (i) $0 \leq p(t) \leq 1, \ \forall t \in [0, 1]$; (ii) $p(t) < \epsilon, \ \forall x \in A$; (iii) $p(t) > 1 - \epsilon, \ \forall t \in B$. [15]

(5) Suppose g is a function having power series expansion:

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with radius of convergence $R > 0$ around $z_0 \in \mathbb{C}$. For $z_1 \in \mathbb{C}$ with $|z_1 - z_0| < R$, find a power series expansion for g around z_1 with suitable radius of convergence. [15]

(6) Compute the Fourier series expansion of the function:

$$f(\theta) = \begin{cases} 5 & \text{if } -\pi \leq \theta \leq 0; \\ -10 & \text{if } 0 < \theta < \pi, \end{cases}$$

extended 2π periodically to $\mathbb{R}$. To which function does the Fourier series of f converge? [15]

(7) Consider

$$l^{\infty}(\mathbb{Z}) = \{f | f : \mathbb{Z} \to \mathbb{C} \text{ with } \sum_{n \in \mathbb{Z}} |f(n)| < \infty\}.$$

For $f, g \in l^{\infty}(\mathbb{Z})$, define $f \star g$ by

$$(f \star g)(n) = \sum_{k \in \mathbb{Z}} f(k) g(n - k).$$

Show that $\star$ is an associative and commutative binary operation on $l^{\infty}(\mathbb{Z})$. Determine as to whether $\star$ operation has an identity (or unit) element. [15]